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Double Diffusive Convection in a Horizontal Porous Layer Superposed by a Fluid Layer

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الانتشار المزدوج في طبقة مسامية تعتليها طبقة مائع

استخدمت طريقة متعددة الحدود لقاعدة شبشف لدراسة الاستقرار من خلال الحصول على نظام من الحلول العددية لطبقتين إحداهما تمثل مائعاً وهي علوية والأخرى تمثل طبقة إسفنجية، وهي سفلية. وقد درس الاستقرار للحل عندما يكون هناك انتقال وذوبان ملحي نتيجة التسخين من اسفل إلى أعلى خلال الطبقتين. وهذه الدراسة تمت عن طريق إيجاد العلاقة بين العدد ريلي الحراري والعدد ريلي الملحي للمناطق الحرجة للطبقة المسامية مع تزايد الحرارة والملوحة، وتم أيضاً تحديد المناطق اللازمة لحدوث الاستقرار من عدمه، والضروري لتعين الحل المطلوب.

In this work, a linear stability analysis, using the spectral Chebyshev polynomial method, is applied to obtain numerically the solution of a multi-layer system consisting of the finger convection onset in a fluid layer overlying a porous layer. The boundary value problem is investigated from the multi-layer problem when the thermal Rayleigh number and the critical salt Rayleigh number for porous layer are increasing due to heating and salting the lower of porous layer. The eigenvalue of the problem, instable and unstable case is obtained.

INTRODUCTION

The boundary value problem of multi-layer tapes records the importance of the work of many authors: Nield (1968) considered the onset of finger convection in a horizontal layer of incompressible viscous fluid of thickness d_f in which a solute is dissolved. Taunton *et al.* (1972) considered the thermohaline instability and salt fingers in a porous medium and solved the boundary value problem. Hilles (1983) and Maples & Poirier (1984) discussed the stability of the boundary value problem of a thermodynamical consistent model, the directional solidification of molten alloys as a layer of porous material of variable permeability which is separated from its melt by a mushy zone of dendrites. Glicksman (1986) described the interaction between the solidifying alloy and its melt by a doubly diffusive model. Chen &Chen (1988) considered the multi-layer problem when the above layer is heated and salted from above, and the solution of its problem is obtained using a shooting method. The same boundary value problem, by using spectral method of Chebyshev polynomial, was solved by the author (1997). Recently, investigating this problem, Chen & Chen (1997) considered a tank with a solute gradient, subjected to lateral heating experimentally. Payne & Straughan (1998) established a continuous dependence result on a critical parameter, which appears in the interfacial boundary conditions. Also, they investigated the Brinkman correction to Darcy's equation and derived a priori convergence result, comparing the solution to that of the Brinkman system of partial differential equation, with that of Darcy equation. Straughan (2001) used the thermal convection in fluid layer overlying a porous layer, which is heated from below. Also, he considered surface tension driven on the free top boundary of upper layer. The same author (2002) dealt with the same two layers system, considering the ratio depth of the relative layer also, investigated the effect of the variation of relevant fluid and porous material properties.

Here, we will discuss the solution of the multi-layer problem, which consists of the onset of finger convection in a fluid layer overlying a porous layer, where the lower layer is heated and salted from below. The directional concentrated alloys solidification contains the mushy zone and consists of dendrites immersed in the melt separate the melt region from the frozen solid region. The spectral Chebyshev polynomial method is used to solve the boundary value problem of the previous multi-layer and its eigenvalue is obtained.

MATHEMATICAL FORMULATION

Consider two horizontal layers L_1 and L_2, where the interface of the bottom layer L_1 with the top layer L_2 is the plane $x_3 = 0$, in Cartesian coordinates x_i, $i = 1,2,3$ the axis is chosen, so that gravity acts in the negative x_3-direction. Consider the layer L_1 of thickness d_f is filled with an incompressible viscous fluid containing a dissolved solute (or salt) while the lower layer L_2 of thickness d_m is a porous medium permeated by the fluid (Fig. 1). Heat is supplied to this configuration so that convection takes place in which temperature driven buoyancy and salting effects are damped by viscosity. One obvious thing that this kind of multi-layer problem occurs when the fluid is at rest and both layers are spanned by temperature and salinity gradients in the x_3-direction. The solution of such boundary value problem, when they flow in the porous layer of thickness d_m, is governed by Darcy's law, while the fluid flow in the upper layer of thickness d_f, governed by the Navier-Stokes equations is called conduction solution.

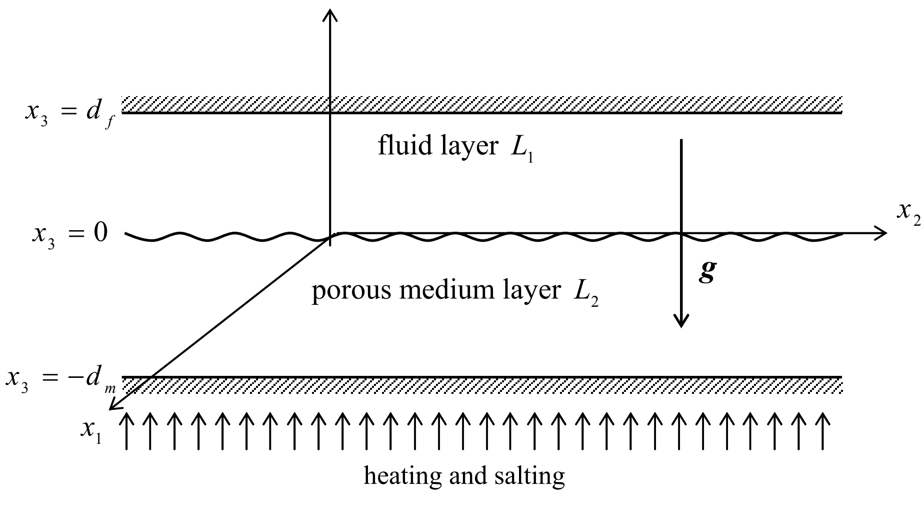

Fig. 1. Diagram of the problem.

Convection is driven by the dependence of the fluid density on temperature and salinity. Typically, the Oberbeck-Boussinesq approximation is made where concepts like local thermal equilibrium, heating from viscous dissipation, radiative effects etc. are ignored as are variation in fluid density, except where they occur in the momentum equation. The fluid density is related to the Kelvin temperature T and salinity S by

$$\rho_f = \rho_0\left[1-\alpha\left(T-T_0\right)+\beta\left(S-S_0\right)\right] \tag{2.1}$$

where ρ_0 is the density at temperature T_0 and salinity S_0, and α, β (both assumed constant) are respectively the thermal and salting coefficients of the volume expansion for the fluid. It is well known that these can be strongly temperature dependent so that (2.1) may be inappropriate[1] for large temperature and salinity variations. Following the approach of Nield (1968) and Chen & Chen (1988), the momentum, energy and salting equations for the flow of an incompressible viscous fluid through a porous medium are

$$\begin{aligned}
\frac{\rho_0}{\phi}\frac{\partial \mathbf{v}_m}{\partial t} &= -\nabla P_m - \frac{\mu}{K}\mathbf{v}_m + \rho_f \mathbf{g} \\
\left(\rho c\right)_m \frac{\partial T_m}{\partial t} + \left(\rho c_p\right)_f \mathbf{v}_m . \nabla T_m &= k_m \nabla^2 T_m \\
\phi \frac{\partial S_m}{\partial t} + \mathbf{v}_m . \nabla S_m &= \mathrm{D}_m \nabla^2 S_m
\end{aligned} \tag{2.2}$$

where the solenoidal vector $\mathbf{v}_m$ denotes fluid seepage velocity, P_m denotes pressure, μ denotes the dynamic viscosity (assumed constant) of the fluid, K and ϕ respectively denote the permeability and porosity of the porous substrate,

George et al. 1989 represent ρ_f by a polynomial of order three in their description of convection in lakes in which the bottom can be represented by a porous layer which is under-pinned by an impermeable permafrost boundary.

ρ_f denotes the fluid density and is given by the formula (2.1), k_m and D_m are respectively the overall thermal conductivity and mass diffusivity of the porous layer, $(\rho c_p)_f$ is the heat capacity per unit volume of the fluid at constant pressure and $(\rho c)_m$ is the over all heat capacity per unit volume of the porous medium at constant pressure. In fact,

$$(\rho c)_m = \phi(\rho c_p)_f + (1-\phi)(\rho c_p)_m$$

where $(\rho c_p)_m$ is the heat capacity per unit volume of the porous substrate.

The top layer L_1 is filled with incompressible Navier-Stokes fluid in which the conservation of momentum, energy and salting are expressed through the equations

$$\rho_0\left(\frac{\partial \mathbf{v}_f}{\partial t} + \mathbf{v}_f.\nabla\mathbf{v}_f\right) = -\nabla P_f + \mu \mathbf{v}_f + \rho_f \mathbf{g}$$

$$(\rho c_p)_f\left(\frac{\partial T_f}{\partial t} + \mathbf{v}_f.\nabla T_f\right) + (\rho c_p)_f \mathbf{v}_f.\nabla T_f = k_f \nabla^2 T_f \qquad (2.3)$$

$$\frac{\partial S_f}{\partial t} + \mathbf{v}_f.\nabla S_f = \mathrm{D}_f \nabla^2 S_f$$

where the solenoidal vector $\mathbf{v}_f$ denotes fluid velocity and k_f and D_f respectively, represent the thermal conductivity and mass diffusivity of the fluid. The Boussinesq approximation has been used in equation (2.2) and (2.3) where the convicted terms in the fluid acceleration have been ignored and Darcy's law has been employed as is customary in the modeling of porous media. The convection problem is completed by the specification of boundary conditions on the upper surface of the viscous fluid layer, at the interface between the fluid and porous layers and at the lower boundary of the porous layer. Many combinations of boundary conditions are possible but for comparison with Chen & Chen (1988), isothermal rigid exterior boundaries are considered giving three conditions on each exterior boundary. Thus

$$
\begin{aligned}
&T_f = T_u, \qquad S_f = S_u, \qquad \mathbf{v}_f = \mathbf{0}, \qquad x_3 = d_f, \\
&T_m = T_l, \qquad S_m = S_l, \qquad \mathbf{v}_m.\mathbf{e}_3 = \mathbf{0}, \qquad x_3 = -d_m.
\end{aligned}
\tag{2.4}
$$

where T_u and T_l are respectively the temperatures at the upper and lower exterior boundaries. At the fluid/porous-medium interface, temperature, heat flux, salinity, salt flux, normal fluid velocity and normal stress are assumed to be continuous. This leaves two final conditions to be specified on the interface. Straughan (2002) has reformulated the two final conditions of Jones (1973) and Beavers-Joseph (1967) in the form,

$$
\frac{\partial u_f}{\partial x_3} + \gamma\ \frac{\partial w}{\partial x_1} = \frac{\alpha_{\mathrm{BJ}}}{\sqrt{K}}\left(u_f - u_m\right), \qquad \frac{\partial v_f}{\partial x_3} + \gamma\ \frac{\partial w}{\partial x_2} = \frac{\alpha_{\mathrm{BJ}}}{\sqrt{K}}\left(v_f - v_m\right), \tag{2.5}
$$

where u_f, v_f are the limiting tangential components of the fluid velocity as the interface is approached from the fluid layer L_1 whereas u_m, v_m are the same limiting components of tangential fluid velocity as the interface is approached from the porous layer L_2 and w is axial velocity. For the constant $\gamma = 1$, in equation (2.5), gives Jones (1973) condition, while $\gamma = 0$, gives Beavers-Joseph (1967) condition. We can easily verify that the field equations (2.2) and (2.3) and all boundary conditions (2.4) are satisfied by the conduction solution

$$
\begin{aligned}
&\mathbf{v}_f = \mathbf{0}, \qquad T_f\big|_E = T_0 + \left(T_u - T_0\right)\frac{x_3}{d_f}, \qquad S_f\big|_E = S_0 + \left(S_u - S_0\right)\frac{x_3}{d_f}, \\
&\mathbf{v}_m = \mathbf{0}, \qquad T_m\big|_E = T_0 + \left(T_0 - T_l\right)\frac{x_3}{d_m}, \qquad S_m\big|_E = S_0 + \left(S_0 - S_l\right)\frac{x_3}{d_m},
\end{aligned}
\tag{2.6}
$$

where the interfacial temperature T_0 and salt concentration S_0 are determined by the continuity of heat flux and salt flux respectively and take the values

$$
T_0 = \frac{k_m d_f T_l + k_f d_m T_u}{k_m d_f + k_f d_m}, \qquad S_0 = \frac{\mathrm{D}_m d_f S_l + \mathrm{D}_f d_m S_u}{\mathrm{D}_m d_f + \mathrm{D}_f d_m}. \tag{2.7}
$$

A hydrostatic pressure accompanies this solution, which is a function of x_3 only.

Following the policy of Nield (1977) and Chen & Chen (1988) displacement and time are rescaled respectively by d_m and d_m^2/λ_m in the porous medium and, by d_f and d_f^2/λ_f in the fluid medium where

$$\lambda_f = \frac{k_f}{(\rho c_p)_f}, \qquad \lambda_m = \frac{k_m}{(\rho c_p)_f}. \tag{2.8}$$

When non-dimensional velocity $\mathbf{u}_m$, temperature θ_m, salinity s_m and hydrostatic pressure p_m are introduced into the porous medium equation (2.2) by the definitions

$$\begin{aligned} \mathbf{v}_m &= \frac{\nu}{d_m}\mathbf{u}_m, & T_m &= T_m\big|_E + \frac{|T_0 - T_l|\nu}{\lambda_m}\theta_m, \\ S_m &= S_m\big|_E + \frac{|S_0 - S_l|\nu}{\mathrm{D}_m}s_m, & P_m &= P_m\big|_E + \frac{\rho_0\nu^2}{K}p_m. \end{aligned} \tag{2.9}$$

Similarly, when non-dimensional velocity $\mathbf{u}_f$, temperature θ_f, salinity s_f and hydrostatic pressure p_f are introduced into the fluid equations (2.3) by the definitions

$$\begin{aligned} \mathbf{v}_f &= \frac{\nu}{d_f}\mathbf{u}_f, & T_f &= T_f\big|_E + \frac{|T_0 - T_u|\nu}{\lambda_f}\theta_f, \\ S_f &= S_f\big|_E + \frac{|S_u - S_0|\nu}{\mathrm{D}_f}s_f, & P_f &= P_f\big|_E + \frac{\rho_0\nu^2}{d_f^2}p_f. \end{aligned} \tag{2.10}$$

perturbation and non-dimensionalisation produce the non-dimensional numbers Da (Darcy number) and G_m which are defined by

$$\mathrm{Da} = \frac{K}{d_m^2}, \qquad G_m = \frac{(\rho c)_m}{(\rho c_p)_f}. \tag{2.11}$$

and $\mathbf{u}_m$ and $\mathbf{u}_f$ are solenoidal vectors and the nondimensional Prandtl numbers Pr_m and Pr_f, Lewis numbers Le_m and Le_f, Rayleigh numbers Ra_m, Ra_f, $\mathrm{Ra}_m^{(s)}$ and $\mathrm{Ra}_f^{(s)}$ are defined by

$$\Pr_m = \frac{\nu}{\lambda_m}, \quad \Pr_f = \frac{\nu}{\lambda_f}, \qquad \mathrm{Le}_m = \frac{\mathrm{D}_m}{\lambda_m}, \quad \mathrm{Le}_f = \frac{\mathrm{D}_f}{\lambda_f},$$
$$\mathrm{Ra}_m = \frac{g\alpha(T_0 - T_l)d_m K}{\nu\lambda_m}, \qquad \mathrm{Ra}_f = \frac{g\alpha(T_u - T_0)d_f^3}{\nu\lambda_f}, \qquad (2.12)$$
$$\mathrm{Ra}_m^{(s)} = \frac{g\beta(S_0 - S_l)d_m K}{\nu\mathrm{D}_m}, \qquad \mathrm{Ra}_f^{(s)} = \frac{g\beta(S_u - S_0)d_f^3}{\nu\mathrm{D}_f}.$$

The linearized approximation of resulting equations (currently exact) is constructed by ignoring all terms involving products of unknown functions.

Normal mode solution is sought for equations, which are obtained, from applying linearized approximate in which all variables $\theta_m, \theta_f, w_m, w_f, s_m, s_f$ are represented by

$$\psi_f = \psi\ (x_3)e^{\sigma_f t}\ e^{i(p_f x_1 + q_f x_2)}.,\ \psi_m = \psi\ (x_3)e^{\sigma_m t}\ e^{i(p_m x_1 + q_m x_2)}.$$

When the fluid and porous momentum equations are treated twice by the curl operator to remove the pressure terms, it follows easily that the fluid layer equations can be recast in the form:

$$\frac{\sigma_f}{\Pr_f}\left(D_f^2 - a_f^2\right)w_f = \left(D_f^2 - a_f^2\right)^2 w_f - a_f^2 \mathrm{Ra}_f \theta_f + a_f^2 \mathrm{Ra}_f^{(s)} s_f,$$
$$\sigma_f \theta_f - H_T w_f = \left(D_f^2 - a_f^2\right)\theta_f, \qquad 0 \le x_3 \le 1 \qquad (2.13)$$
$$\frac{\sigma_f}{\mathrm{Le}_f} s_f - H_S w_f = \left(D_f^2 - a_f^2\right)s_f,$$

where the wave number of the fluid layer is given by $a_f^2 = p_f^2 + q_f^2$. The porous medium equations then become

$$\left(\frac{\mathrm{Da}}{\Pr_m}\frac{\sigma_m}{\phi} + 1\right)\left(D_m^2 - a_m^2\right)w_m = -a_m^2 \mathrm{Ra}_m \theta_m + a_m^2 \mathrm{Ra}_m^{(s)} s_m,$$
$$G_m \sigma_m \theta_m - H_T w_m = \left(D_m^2 - a_m^2\right)\theta_m, \qquad -1 \le x_3 \le 0 \qquad (2.14)$$
$$\frac{\sigma_m \phi}{\mathrm{Le}_m} s_m - H_S w_m = \left(D_m^2 - a_m^2\right)s_m,$$

where $H_T = \text{sign}(T_l - T_0) = \text{sign}(T_0 - T_u)$, $H_S = \text{sign}(S_l - S_0) = \text{sign}(S_0 - S_u)$. and the wave number of the porous layer is given by $a_m^2 = p_m^2 + q_m^2$.

From these equations, we have

$$\begin{aligned} D_f\psi &= \frac{d\psi}{dx_3} \quad (0 < x_3 < 1), & a_f &= \hat{d} a_m, \\ D_m\psi &= \frac{d\psi}{dx_3} \quad (-1 < x_3 < 0), & \sigma_f &= \frac{\hat{d}^2}{\varepsilon_T}\sigma_m. \end{aligned} \tag{2.15}$$

$$w_f = D_f w_f = \theta_f = s_f = 0 \qquad \text{on } x_3 = 1. \tag{2.16}$$

$$\left.\begin{aligned} & w_f = \hat{d} w_m, \quad \gamma_T\theta_f = \varepsilon_T\theta_m, \quad \gamma_S s_f = \varepsilon_S s_m, \\ & D_f\theta_f = \varepsilon_T D_m\theta_m, \quad D_f s_f = \varepsilon_S D_m s_m, \\ & D_f^3 w_f - 3a_f^2 D_f w_f + \frac{\hat{d}^4}{\text{Da}} D_m w_m = \frac{\sigma_f}{\text{Pr}_f} D_f w_f - \frac{\hat{d}^4\sigma_m}{\phi\,\text{Pr}_m} D_m w_m, \\ & \frac{\hat{\alpha}\hat{d}}{\sqrt{\text{Da}}}\left[D_f w_f - \hat{d}^2 D_m w_m\right] = D_f^2 w_f. \end{aligned}\right\} x_3 = 0. \tag{2.17}$$

$$w_m = \theta_m = s_m = 0 \qquad \text{on } x_3 = -1. \tag{2.18}$$

where the parameters $\varepsilon_T, \varepsilon_S, \gamma_T$ and γ_S are defined by

$$\varepsilon_T = \frac{\lambda_f}{\lambda_m}, \qquad \varepsilon_S = \frac{\text{D}_f}{\text{D}_m}, \qquad \gamma_\text{T} = \frac{T_u - T_0}{T_0 - T_l}, \qquad \gamma_\text{S} = \frac{S_u - S_0}{S_0 - S_l}. \tag{2.19}$$

In the next section we turn to the method of solution.

METHOD OF SOLUTION

Let the variables $y_1, y_2, \ldots, y_{14}$ be defined as:

$$
\begin{aligned}
&y_1 = w_f, && y_2 = D_f w_f, && y_3 = D_f^2 w_f, && y_4 = D_f^3 w_f, \\
&y_5 = \theta_f, && y_6 = D_f \theta_f, && y_7 = s_f, && y_8 = D_f s_f, \\
&y_9 = w_m, && y_{10} = D_m w_m, && y_{11} = \theta_m, && y_{12} = D_m \theta_m, \\
&y_{13} = s_m, && y_{14} = D_m s_m.
\end{aligned} \tag{3.1}
$$

Then it is straightforward to verify that equation (2.13) can be rewritten as in the form of:

$$D_f y_1 = y_2, \; D_f y_2 = y_3, \; D_f y_3 = y_4,$$

$$D_f y_4 = (2a_f^2 + \frac{\sigma_f}{\Pr_f}) y_3 - (a_f^4 + \frac{\sigma_f a_f^2}{\Pr_f}) y_1 + a_f^2 Ra_f y_5 - a_f^2 Ra_f^{(s)} y_7, \tag{3.2}$$

$$D_f y_5 = y_6, \; D_f y_6 = (a_f^2 + \sigma_f) y_5 - y_1,$$

$$D_f y_5 = y_6, \; D_f y_8 = (a_f^2 + \sigma_f / L\rho_f) y_7 - y_1$$

Also the formula (2.14) takes the form of:

$$D_f y_9 = y_{10},$$

$$(1 + \frac{\sigma_f Da}{\Pr_f \phi}) D_m y_{10} = (a_m^2 + \frac{\sigma_m Da}{\Pr_m \phi}) y_9 - a_m^2 Ra_m y_{11} + a_m^2 Ra_m^{(s)} y_{13}, \tag{3.3}$$

$$D_m y_{11} = y_{12} \;, \; D_m y_{12} = (a_m^2 + \sigma_m G_m) y_{11} - y_9$$

$$D_m y_{13} = y_{14}, \; D_m y_{14} = (a_m^2 + \sigma_m \phi / L\rho_m) y_{13} - y_9 .$$

The formula (3.2) and (3.3) can be adapted respectively in the following form:

The two formulas (3.4) and (3.5) represent a system of partial differential equations, where the values of y_1, y_2, y_3, y_4 will be obtained directly with the aid of (3.4), while y_9, y_{11}, y_{13} will be discussed with the aid of (3.5).

Then the linear system of partial differential equations of (3.4) and (3.5) will be solved under the following conditions

$$y_1(1) = y_2(1) = y_5(1) = y_7(1) = y_9(-1) = y_{11}(-1) = y_{13}(-1) = 0, \quad \text{and} \quad (3.6)$$

$$y_6(-1) + \varepsilon_T y_{12}(1) = 0, \; y_8(-1) + \varepsilon_s y_{14}(1) = 0$$

$$-y_4(-1) + 3a_f^2 y_2(-1) - \frac{\hat{d}^4}{Da} y_{10}(1) = \sigma y\left(-\frac{\hat{d}^2}{\varepsilon_T \Pr_f} y_2(-1) + \frac{\hat{d}^2}{\phi \Pr_m} y_{10}(1)\right),$$

$$\hat{d}\alpha_{Bf} y_2(-1) - \sqrt{Da}\, y_3(-1) - \hat{d}^3 \alpha_{Bf} y_{10}(1) = 0 \qquad (3.7)$$

To obtain the solution of equations (3.4) and (3.5) under the conditions (3.6) and (3.7), we will use the spectral method of Chebyshev polynomial. The expansions in Chebyshev polynomials with its orthogonal and spectral relations are better suited to the solutions of hydrodynamic stability problems than expansions in other sets of orthogonal polynomials. Results of high accuracy are obtained when using Chebyshev approximation to compute solution of boundary value problems (see Table 1). For this, let $y(z)$, the unknown function be expanded in the form of

$$y(z) = \sum_{n=0}^{\infty} a_n T_n(z) \qquad (3.8)$$

where $T_n(z)$ are the Chebyshev polynomials of the first kind of order n, $n \geq 0$, and a_n are constants, represent the coefficients of the Chebyshev polynomials. The

Chebyshev polynomials satisfy the following orthogonal relation of Ergeyli & Linz (1985):

$$\int_{-1}^{1} \frac{T_m(z)T_n(z)}{\sqrt{1-z^2}} dz = \begin{cases} 0 & n \neq m \\ \pi/2 & n = m \neq 0 \\ \pi & n = m = 0 \end{cases} \quad (3.9)$$

This relation enables us to evaluate the coefficients a_n, in the form

$$a_n = \frac{2}{\pi} \int_{-1}^{1} \frac{y(z)T_n(z)}{\sqrt{1-z^2}} dz, n \neq 0$$

$$a_o = \frac{1}{\pi} \int_{-1}^{1} \frac{y(z)}{\sqrt{1-z^2}} dz \quad (3.10)$$

To represent the first and higher derivatives of Eq. (3.8) in the Chebyshev polynomials, we find

$$\frac{dy}{dz} = \sum_{n=0}^{\infty} a_n T_n(z)$$

(3.11)

where $T_n'(z)$ is a polynomial of degree (n-1). Now represent $T_n'(z)$ as an expansion of Chebyshev polynomial, i. e

$$T_n'(x) = \sum_{s=0}^{\infty} b_{s,n} T_s(x), \qquad b_{s,n} = 0 \quad \text{if} \quad s \geq n \quad (3.12)$$

The values of $b_{s,n} = 0$ can be obtained, directly, by using the orthogonal relation (3.9), as

$$b_{s,n} = \frac{2}{\pi}\int_{-1}^{1}\frac{T_n'(z)T_s(z)}{\sqrt{1-z^2}}dz \qquad s \geq 1,$$

$$b_{o,n} = \frac{1}{\pi}\int_{-1}^{1}\frac{T_n'(z)}{\sqrt{1-z^2}}dz \qquad s = 0 \qquad (3.13)$$

Using the definition. $T_n(z) = \cos(n\cos^{-1} z)$, in equation (3.13) the values of $b_{s,n}$, $b_{0,n}$ for any value of s and n can be calculated, hence, equation (3.12)can be adapted in the following form:

$$T_{2m}'(z) = 4m\sum_{j=1}^{\infty} T_{2j-1}(z)$$

$$T_{2m+1}'(z) = (2m+1)[1 + 2\sum_{j=1}^{\infty} T_{2j}(z)] \qquad (3.14)$$

=

Also higher derivative of $T_n(z)$ can be obtained, using the recurrence relation for equation (3.12) and (3.14). Hence, using (3.12) in (3.13) in (3.11), we get

$$\frac{dy}{dz} = \sum_{n=o}^{\infty}\ \sum_{n=0}^{\infty} b_{j,n} a_n T_j(z) \qquad (3.15)$$

where the values of $b_{j,n}$ can be determined by the two relations of equation (3.13), while a_n are the unknown eigenvalues. The higher derivatives of $y(z)$ can be obtained in the form of

Chebyshev polynomials as

$$\frac{d^r y}{dz^r} = \sum_{n=o}^{\infty}\ \sum_{n=0}^{\infty} b^r{}_{j,n} a_n T_j(z) \qquad (3.16)$$

where the matrix b is called the derivative matrix. The all coefficients of $b_{j,n}^{(r)}$ can be obtained equation (3.13).

In practice, we find that are cannot use the infinite series of Chebyshev polynomials, and we have to rely on a finite approximation of suitable accuracy, so we look for an approximation of the form

$$Y(z) = \sum_{n=0}^{N-1} a_n T_n(z) \, , N = 1,2,...,14. \qquad (3.17)$$

Adapting the general formula of the differential equation of (3.16) in the form of finite series of (3.17), then let $D_f = D_m = D$ and put $\sigma_m = \sigma$ which leads us to get $\sigma_f = \frac{\hat{d}^2}{\varepsilon_T}\sigma$, hence the eigenvalue problem of (3.4) and (3.5), under the adapting condition of (3.6) and (3.7), can be reformulated in the form

$$\frac{dY}{dz} = AY + \sigma BY, \quad z \in [-1,1], \qquad (3.18)$$

where A and B are real 14×14 matrices. The final eigenvalue problem of equation (3.18) reduces to the form

$$EV = \sigma FV \qquad (3.19)$$

where the two matrices E and F have the block forms.

RESULTS

In this section, we will discuss, according to the numerical results, the stability of the solution of multi-layer problem, when heated and salted from below for isothermal rigid boundaries under certain conditions (see section 3).

This result, as in Table 1, is obtained, when the initial Rayleigh number of a porous medium is 50 and the other values and numbers, that are Beavers-Joseph constant $\alpha_{BJ} = 0.1, \sqrt{\text{Da}} = 0.003, \varepsilon_T = 0.7, \varepsilon_S = 3.75$. Also Chen & Chen (1988) results

$$\text{Le}_f = \frac{\text{D}_f}{\lambda_f}, \qquad \text{Le}_m = \frac{\text{D}_m}{\lambda_m}, \qquad \text{Pr}_{fs} = \frac{\text{Pr}_f}{\text{Le}_f},$$

$$\text{Pr}_{ms} = \frac{\text{Pr}_m}{\text{Le}_m}, \qquad \gamma_T = \frac{\hat{d}}{\varepsilon_T}, \qquad \gamma_S = \frac{\hat{d}}{\varepsilon_S}.$$

Further, the numerical results are obtained, after writing the critical Rayleigh number and the corresponding wave number as extension of corresponding minimum marginal stable curve, where

$$Ra_m^{(s)} = Ra_m^{(s)}(Ra_f^{(s)}, Ra_m, Ra_f, \text{Pr}_m, \text{Pr}_f),$$

and

$$a_m = a_m(Ra_f^{(s)}, Ra_m, Ra_f, \text{Pr}_m, \text{Pr}_f).$$

Consider different values of the depth $\hat{d}$, which represents the ratio between the depth of fluid layer d_f to the depth of porous layer d_m, then calculate the corresponding values of the critical salt Rayleigh numbers $\text{Ra}_m^{(s)}$ and the critical wave number a_m, the stability of the boundary value problem is discussed in Table 1. The following remarks can be discussed.

Remark 1. If the depth ratio is around $\hat{d} \leq 0.005$, for the corresponding values of $\text{Ra}_m^{(s)}$ and a_m, we have a stable system, and the eigenvalues of the boundary value problem are real.

Remark 2. For about $0.01 \le \hat{d} \le 0.15$, the corresponding discussion of the solution is overstable and the eigenvalues in this case, are complex.

Remark 3. For $0.15 \le \hat{d} \le 0.2$, we have a stable case and the corresponding eigenvalues for boundary value problem are real.

Table 1. The critical value of $\mathrm{Ra}_m^{(s)}$ and a_m for different depth ratio $\hat{d}$.

$\mathrm{Ra}_m^{(s)}$	$\hat{d}$	a_m	System state
696.2762	0.2	2.7187	stable
383.7989	0.17	3.1718	stable
184.3388	0.15	3.6403	Overstable
107.8987	0.10	3.7500	Overstable
78.4814	0.05	3.7500	Overstable
70.4207	0.04	3.7500	Overstable
12.2648	0.01	2.4000	Overstable
8.2643	0.005	2.4000	stable

The relation between the values of $\mathrm{Ra}_m^{(s)}$ and a_m, for fixed $\hat{d}$ are obtained in Fig. 1. Hence the following remarks can be written.

Remark 4. For values of $\hat{d} = 0.01$, $\hat{d} = 0.05$, $\hat{d} = 0.75$ and $\hat{d} = 0.1$, the corresponding Rayleigh numbers have very small values, in the range of wave numbers from $a_m = 0$ to about $a_m = 2.5$, (overstable case).

Remark 5. For $\hat{d} = 0.15$ and $\hat{d} = 0.2$, the values of the Rayleigh number increases according to the increase in wave number, (stable case).

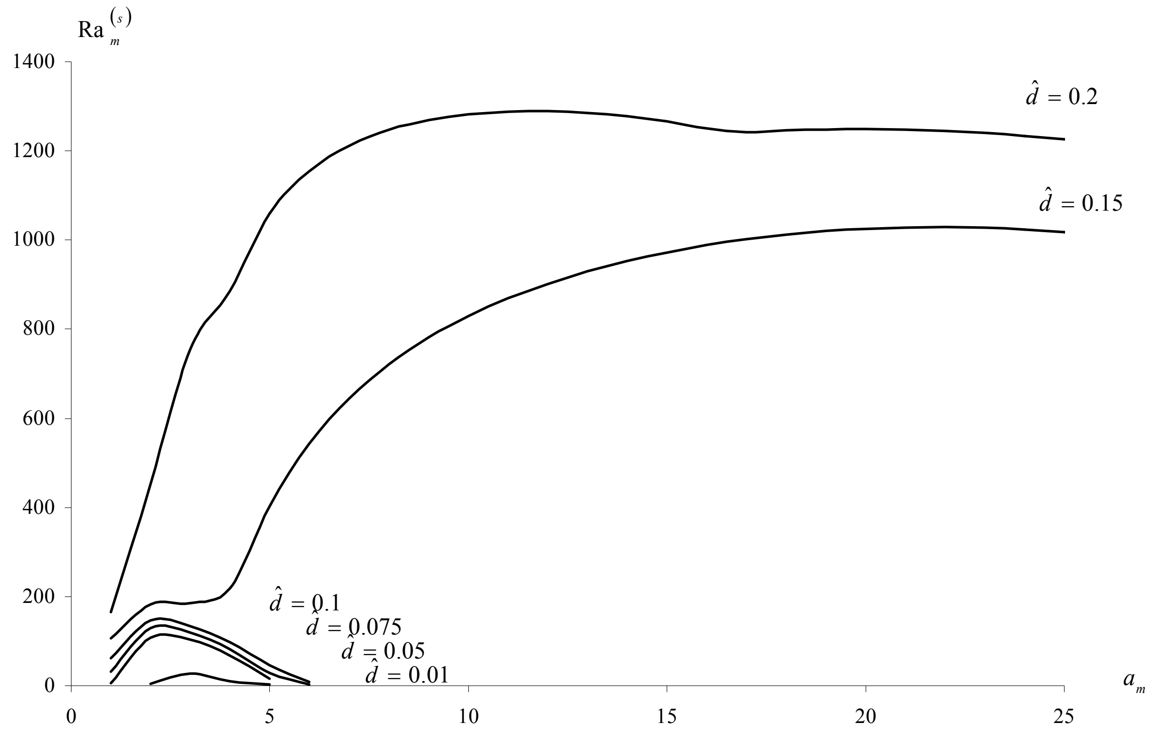

Fig. 1. The relation between $\text{Ra}_m^{(s)}$ and a_m .

CONCLUSIONS

From the previous discussion we follow that:

1- The problem of the onset of finger convection in a horizontal porous layer superposed by fluid layer has been investigated. This system is salted and heated from below. The boundary conditions are four on the top boundary of the top Layer, but the lower boundaries of the lower Layer are three conditions. While the interface conditions are seven. The spectral Chebyshev polynomial method has been used for solving the eigenvalue problem.

2- For the depth ratio $\hat{d}$ between the values 0.01 to 0.15, and the initial thermal Rayleigh number $\mathrm{Ra}_m = 50$, the eigenvalues have complex number with positive real part. This indicates that the system is overstable, otherwise, the system is instable.

3- The value of critical Rayleigh number $\mathrm{Ra}_m^{(s)}$ increases when the depth ratio $\hat{d}$ is increased and the stable cases are obtained.

4- The marginal curves between $\mathrm{Ra}_m^{(s)}$ and a_m is illustrated.

5- The spectral Chebyshev polynomial method is considered as one of the best methods for discussing the stability of the boundary value problem of multi-layer cases.

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