Triangular norm based predicate fuzzy logics

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Abstract

The paper surveys the present state of knowledge on t-norm based predicate fuzzy logics with their double semantics: standard (the set of truth values being the real interval [0,1]) and general with abstract algebras of truth functions. © 2009 Published by Elsevier B.V.

Keywords: Mathematical fuzzy logic; Predicate logics; t-Norms; Standard semantics; Arithmetical hierarchy; Core fuzzy logics; Model theory

0. Introduction

Mathematical fuzzy logic, understood as a part of mathematical (symbolic) logic, studies systems of many-valued logics with a comparative notion of truth and the real unit interval as the intended (standard) set of truth values. The metamathematics of both propositional and predicate logics of this kind was developed in [66] for the case of continuous t-norm based logics and generalized in [42] for the case of left-continuous t-norms. Since then there has been a substantive development3 with many authors involved (see the references or recent surveys [63,64]). This paper is a survey of results on predicate (first-order) t-norm based fuzzy logics. We concentrate on technical mathematical results and neglect motivational issues like relation of mathematical fuzzy logic and theories of vagueness, see e.g., papers [49,82] and books [101,132,134].

By the term ‘t-norm based logics’ we mean fuzzy logics determined by the semantics of real unit interval with conjunction interpreted by a left-continuous t-norm (and implication by its residuum), i.e., our conjunction is always commutative and 1 is its unit, in proof-theoretic terms we restrict ourselves to logics with the structural rules of exchange and weakening. We start with the weakest t-norm based fuzzy logic MTL and concentrate on its most notable extensions like the Hájek’s BL, Łukasiewicz, Gödel, and Product logics (see the monograph [66]). We shall study these logics in

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3 It led to establishment of the working group on Mathematical Fuzzy Logic www.mathfuzzlog.org with, currently, more than 90 members.

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Table 1
Some usual axiom schemata in fuzzy logics.

<table>
<thead>
<tr>
<th>Axiom schema</th>
<th>Name</th>
</tr>
</thead>
<tbody>
<tr>
<td>¬¬φ → φ</td>
<td>Involution (Inv)</td>
</tr>
<tr>
<td>¬φ ∨ ((φ → φ&amp;ψ) → ψ)</td>
<td>Cancellation (Can)</td>
</tr>
<tr>
<td>¬(φ&amp;ψ) ∨ ((ψ → φ&amp;ψ) → φ)</td>
<td>Weak Cancellation (WCan)</td>
</tr>
<tr>
<td>φ → φ&amp;ψ</td>
<td>Contraction (C)</td>
</tr>
<tr>
<td>φ &amp; ¬φ → ⊤</td>
<td>Pseudocomplementation (PC)</td>
</tr>
<tr>
<td>φ &amp; ψ → φ&amp;((φ → ψ) &amp; ψ)</td>
<td>Divisibility (Div)</td>
</tr>
<tr>
<td>(φ&amp;ψ → ⊤) ∨ (φ &amp; ψ → φ&amp;ψ)</td>
<td>Weak nilpotent minimum (WMN)</td>
</tr>
<tr>
<td>φ ∨ ¬φ</td>
<td>Excluded middle (EM)</td>
</tr>
</tbody>
</table>

Table 2
Some axiomatic extensions of MTL obtained by adding the corresponding additional axiom schemata.

<table>
<thead>
<tr>
<th>Logic</th>
<th>Additional axiom schemata</th>
<th>References</th>
</tr>
</thead>
<tbody>
<tr>
<td>SMTL</td>
<td>(PC)</td>
<td>[68]</td>
</tr>
<tr>
<td>PMTL</td>
<td>(Can)</td>
<td>[68]</td>
</tr>
<tr>
<td>WCMTL</td>
<td>(WCan)</td>
<td>[112]</td>
</tr>
<tr>
<td>IMTL</td>
<td>(Inv)</td>
<td>[42]</td>
</tr>
<tr>
<td>WNM</td>
<td>(WNM)</td>
<td>[42]</td>
</tr>
<tr>
<td>NM</td>
<td>(Inv) and (WNM)</td>
<td>[42]</td>
</tr>
<tr>
<td>BL</td>
<td>(Div)</td>
<td>[66]</td>
</tr>
<tr>
<td>SBL</td>
<td>(Div) and (PC)</td>
<td>[43]</td>
</tr>
<tr>
<td>Ł</td>
<td>(Div) and (Inv)</td>
<td>[66,106]</td>
</tr>
<tr>
<td>II</td>
<td>(Div) and (Can)</td>
<td>[85]</td>
</tr>
<tr>
<td>G</td>
<td>(C)</td>
<td>[66,39,57]</td>
</tr>
<tr>
<td>CPC</td>
<td>(EM)</td>
<td></td>
</tr>
</tbody>
</table>

the broader context of core and △-core fuzzy logics. These two classes were introduced in our paper [84], provide a suitable level of abstraction to formulate our general definitions and results, and contain all the t-norm based fuzzy logics mentioned in [63,64]. After their formal definition in Section 1.1 we will see that:

- The following logics known from the literature are core fuzzy logics in the sense of Definition 1.4: BL, MTL, IMTL, IMTL, WCMTL NM, WNM, Łukasiewicz logic, product logic, and Gödel logic, RPL (rational Pavelka logic), SBL, PL and extensions of the majority of these logics by truth constants (for their definitions see Table 2 or papers [41–43,46,66,86,97,131]).

- The following logics known from the literature are △-core fuzzy logics in the sense of Definition 1.4: extensions of all logics mentioned above by △, SBL, II, G, PL, LII, LII, and RŁII (for their definitions see [2,43,44,66,97]) (Table 1).

As we said, all these logics also appear in survey [64] (with exception of some logics with rational constants introduced recently in [41,46]). Thus the reader can ignore the preliminary section on (△-core fuzzy logics) and read the following sections keeping his/her favorite fuzzy logic in mind.

The paper is organized as follows: Section 1 contains definitions of propositional (△-)core and t-norm based fuzzy logics. Section 2 introduces basic notions of the predicate fuzzy logic: syntax, semantics (primarily an algebraic one but also a sample of game game-theoretic one), completeness theorem, and surveys basic properties of particular predicate fuzzy logics (like BL or Ł) not covered in next sections. Section 3 presents several different possible variants of predicate t-norm based fuzzy logics (in particular logics with crisp equality and logics of wider or narrower semantics)

Following [32] we use the name ‘△-core fuzzy logics’ rather than the original name ‘△-fuzzy logics’ of [84].

There is a much more general formal notion of fuzzy logic introduced in [31] and refined in [37] which includes all (△-)core fuzzy logics. Some of the results we present here hold for those general fuzzy logics as well.

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and we also study fragments of these logics. Section 4 studies positions of several classes of formulas induced by our logics within the arithmetical hierarchy, Section 5 samples model-theoretical results achievable in our logics, and Section 6 shows examples of important theories over predicate fuzzy logics, like variants of Peano arithmetics, Zermelo–Fraenkel-style set theory, and Cantor-style set theory with full comprehension.

As stated above, we present basic definitions and results and then some advanced topics. Our goal is to collect the majority of (mathematically) interesting and promising results in predicate t-norm based fuzzy logic. Because this survey was originally supposed to be published in 2006 (and also, due to the space restriction and personal preferences of the authors) many interesting results are just mentioned and referred to and several interesting and recently studied areas of predicate mathematical fuzzy logic were left-out completely, namely:

- proof-theoretic aspects of predicate fuzzy logics [5,7,9,27]. This subject is, however, thoroughly treated in the recent monograph [108] by Metcalfe, Olivetti and Gabbay.
- fuzzy logics lacking some further structural rules, e.g., exchange or weakening. These logics clearly cannot be based on t-norms, but sometimes they are well-related to some ‘generalized’ notion of t-norm: in particular the logics of pseudo-t-norms (non-commutative ones) were studied in [69,73,92] and logic of uninorms (operation with unit different from 1) were developed in [107].
- the study of semantically given family of Gödel logics as conducted by the Vienna group headed by Baaz [8,4,3].

Given a subalgebra $V$ of the standard Gödel algebra they define logic $G(V)$ but for many choices of $V$ the resulting logic is not a core fuzzy logic (as the deduction theorem ceases to be valid). Axiomatizability and arithmetical complexity of these logics is studied in [6,81,83,75].
- the logic with evaluated syntax studied by Vilém Novák and his group [124,118]. They replace formulas of Łukasiewicz by the so-called evaluated (signed, labeled) formulas, i.e., pairs formula-truth degree. The variant of this logic based on rational-numbers-based [121] is faithfully mutually interpretable with the core fuzzy logic RPL [see the end of Section 2.4] but it can be seen as a more elegant means of proving partially true conclusions from fuzzy sets of axioms. 6

However, the areas mentioned above are either covered by their own monographs (the first and the fourth one) or are, in fact, out of the scope of this paper (t-norm based fuzzy logics). Thus we hope that this paper is a well-rounded representative survey to the growing area of the predicate fuzzy logics.

## 1. Preliminaries—propositional logics

We assume the reader to be familiar with propositional fuzzy logics at least in the level of the survey papers [63,64]. In Section 1.1 below we recall the definition of the logic MTL, (Δ-)core based fuzzy logics and in Section 1.2 we propose a definition of t-norm based fuzzy logics.

In this paper we meet different propositional languages thus we need to recall one important general definition. By an $\mathcal{L}$-theory we mean just a set of formulas in the propositional language $\mathcal{L}$. We do not explicitly mention what is a logic (as a mathematical object) for us, thus the reader can keep his/her favorite notion in mind. However by $\vdash_\mathcal{L}$ we always mean the provability relation between sets of formulas and formulas induced by the logic $\mathcal{L}$.

**Definition 1.1.** Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be propositional languages and $\mathcal{L}_i$ be a logic in $\mathcal{L}_i$. We say $L_2$ is an expansion of $L_1$ if for each $\mathcal{L}_1$-theory $T$ and each $\mathcal{L}_1$-formula $\varphi$ holds that $T \vdash_{\mathcal{L}_1} \varphi$ implies $T \vdash_{\mathcal{L}_2} \varphi$. The extension is conservative if also $T \vdash_{\mathcal{L}_2} \varphi \implies T \vdash_{\mathcal{L}_1} \varphi$. We say that $L_1$ is the $\mathcal{L}_1$-fragment of $L_2$ if $L_2$ is a conservative expansion of $L_1$. In this case we denote $L_1$ by $L_2 \upharpoonright L_1$.

If $\mathcal{L}_1 = \mathcal{L}_2$ we call $L_2$ an extension of $L_1$ instead of expansion. We say that this expansion is axiomatic if there is a set of axioms $\mathcal{A}$ such that $L_2 = L_1 + \mathcal{A}$.

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6 Let us also mention in passing a general approach of Gerla [55] which also works evaluated syntax and is no more t-norm based and even not necessarily truth functional (not using truth functions of connectives). This is surely an interesting and justified possible approach but lies outside of the scope of our paper.

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1.1. (Δ-)core Fuzzy logics

The logic MTL was introduced by Esteva and Godo in [42]. It has three basic binary connectives →, ∧ and & and a nullary connective 0. Further connectives are defined as follows:

\[ \varphi \lor \psi \text{ is } ((\varphi \to \psi) \to \psi) \land ((\psi \to \varphi) \to \varphi), \]

\[ \neg \varphi \text{ is } \varphi \to 0, \]

\[ \varphi \equiv \psi \text{ is } (\varphi \to \psi) \land (\psi \to \varphi). \]

As a general rule that unary connectives are the most binding connectives (operations), implication and equivalence are the least binding ones, and the remaining binary one lie between these two extremes. The following formulas are axioms of MTL:

(A1) \( \varphi \to (\psi \to \chi) \to (\varphi \to \chi) \),

(A2) \( \varphi \& \psi \to \varphi \),

(A3) \( \varphi \& \psi \to \psi \& \varphi \),

(A4a) \( \varphi \& (\varphi \to \psi) \to \varphi \land \psi \),

(A4b) \( \varphi \land \psi \to \varphi \),

(A4c) \( \varphi \land \psi \to \psi \land \varphi \),

(A5a) \( (\varphi \to (\psi \to \chi)) \to (\varphi \& \psi) \to \chi \),

(A5b) \( (\varphi \& \psi) \to \chi \to (\varphi \to (\psi \to \chi)) \),

(A6) \( ((\varphi \to \psi) \to \chi) \to ((\psi \to \varphi) \to \chi) \),

(A7) \( 0 \to \varphi \).

The deduction rule is modus ponens (from \( \varphi \) and \( \varphi \to \psi \) infer \( \psi \)). Observe that whereas the axioms corresponding to the structural rules of weakening and exchange are provable in MTL ((A2) and (A3)), the axiom corresponding to contraction, \( \varphi \to \varphi \& \varphi \) is not provable in MTL. Thus MTL can be viewed as a contraction-free substructural logic. It can be shown that MTL is an extension of Full Lambek Calculus with exchange and weakening [52] (also known as multiplicative additive fragment of the affine intuitionistic linear logic [56] or Höhle’s monoidal logic [94]) by axiom (A6). One can also show that adding the contraction axiom to the MTL would lead to the well known Gödel–Dummett logic [39]. Table 2 contains the most important axiomatic extensions of MTL.

The logic MTL\(_\Delta\) was introduced in [42]. It is the expansion of the logic MTL by a new unary connective \( \Delta \) (called Baaz delta [2]) the deduction rule of necessitation (from \( \varphi \) infer \( \Delta \varphi \)) and the following axioms:

(AΔ1) \( \Delta \varphi \lor \neg \Delta \varphi \),

(AΔ2) \( \Delta (\varphi \lor \psi) \to \Delta \varphi \lor \Delta \psi \),

(AΔ3) \( \Delta \varphi \to \varphi \),

(AΔ4) \( \Delta \varphi \to \Delta \varphi \),

(AΔ5) \( \Delta (\varphi \to \psi) \to (\Delta \varphi \to \Delta \psi) \).

Basic logical notions of proof, theorem, etc. are defined as usual. One of the consequences of the non-idempotency of conjunction is the failure of the classical deduction theorem MTL. However, we can prove a weaker variant in MTL and a different one in MTL\(_\Delta\). To do that define \( \varphi^0 = 1 \) and for each natural \( n > 0 \) define \( \varphi^n = \varphi^{n-1} \& \varphi \).

**Lemma 1.2.** The logic MTL enjoys the Local Deduction Theorem (**LDT**): if for each theory \( T \) and formulas \( \varphi, \psi \) holds: \( T, \varphi \vdash \psi \) iff there is natural number \( n \) such that \( T^T \varphi^n \to \psi \).

The logic MTL\(_\Delta\) enjoys the Delta Deduction Theorem (**DNT**): for each theory \( T \) and formulas \( \varphi, \psi \) it holds that \( T, \varphi \vdash \psi \) iff \( T^T \Delta \varphi \to \psi \).

**Lemma 1.3** (Substitution rule). For all formulas \( \varphi, \psi, \) and \( \chi \) of MTL (or MTL\(_\Delta\)) we can prove than in the respective logic:

\[ \varphi \equiv \psi \vdash \chi(\varphi) \equiv \chi(\psi). \] (1)

These two properties (the deduction theorems and validity of the substitution rule) are the crucial properties of MTL and MTL\(_\Delta\) and we define a wide class of (fuzzy) logics sharing these properties.
Definition 1.4. Let \( L \) be a logic in the language \( \mathcal{L} \) containing that of MTL such that it satisfies (1) for each formulas \( \varphi, \psi, \) and \( \chi \) of the language \( \mathcal{L} \). We say that \( L \) is a core fuzzy logic if

- \( L \) expands MTL,
- \( L \) has \( \mathcal{L}DT \).

We say that \( L \) is a \( \Delta \)-core fuzzy logic if

- \( L \) expands MTL\(_\Delta \),
- \( L \) has \( \mathcal{D}T \_\Delta \).

If we want to say that a logic is either core of \( \Delta \)-core fuzzy logic we simply write that it is a \( (\Delta) \)-core fuzzy logic. The congruence condition together with the ‘expanding conditions’ entail that each \( (\Delta) \)-core fuzzy logic is an implicational logic in the sense of Rasiowa (see [127]). The following lemma is the direct consequence of [31, Corollary 8, Theorem 6].

Lemma 1.5. The logic \( L \) which satisfies (1) and expands MTL, is a core fuzzy logic iff it is axiomatizable by an axiomatic system with modus ponens as the only deduction rule.

The logic \( L \) which satisfies (1) and expands MTL\(_\Delta \), is a \( \Delta \)-core fuzzy logic iff it is axiomatizable by an axiomatic system with modus ponens and necessitation as the only deduction rules.

Thus each axiomatic extension of MTL a core fuzzy logic and each axiomatic extension of MTL\(_\Delta \) is a \( \Delta \)-core fuzzy logic. In particular, MTL is a core fuzzy logic and MTL\(_\Delta \) is \( \Delta \)-core fuzzy logic (MTL\(_\Delta \), is not a core fuzzy logic—we need to add a new deduction rule). All logics from Table 2 are core fuzzy logics, for other examples of \( (\Delta) \)-core fuzzy logics see the beginning of this section.

Corollary 1.6. Let \( L \) be \( (\Delta) \)-core fuzzy logic, \( T \) a theory, and \( \varphi, \psi, \chi \) formulas. Then the following holds:

- Proof by cases Property (PCP): if \( T, \varphi \vdash \chi \) and \( T, \psi \vdash \chi \), then \( T, \varphi \lor \psi \vdash \chi \).
- Semilinearity Property (SLP): if \( T, \varphi \rightarrow \psi \vdash \chi \) and \( T, \psi \rightarrow \varphi \vdash \chi \), then \( T \vdash \chi \).

The validity of both these properties can be easily demonstrated using the deduction theorems and it is one of the main reasons for our choice of the class of \( (\Delta) \)-core fuzzy logics (see [31] for more details). In fact, the Semilinearity Property is used [37] as the definition of the very wide class of the so-called weakly p-implicational semilinear logics\(^7\) which contain all \( (\Delta) \)-core fuzzy logics.

Now we recall basic semantical notions. We introduce a notion of \( L \)-algebra. The reader familiar with Abstract Algebraic Logic (AAL, see [51] for a recent survey) easily observes that \( (\Delta) \)-core fuzzy logics are algebraizable (in the sense of Blok and Pigozzi, see [25]) with the translation sets \( \Delta(p, q) = \{ p \equiv q \} \) and \( E(p) = \{ p \approx 1 \} \). Thus we know that our logics are sound and complete w.r.t. their corresponding equivalent algebraic semantics. However, in fuzzy logic we are interested in completeness w.r.t. some particular subclass of the corresponding algebras, e.g., the linearly ordered ones and the so-called standard ones (those with domain [0, 1], see Section 1.2).

Definition 1.7 (Esteva and Godo [42]). An MTL-algebra is a structure \( B = (B, \lor, \land, \& , \rightarrow, 0, 1) \) such that:

1. \( (B, \lor, \land, 0, 1) \) is a bounded lattice,
2. \( (B, \& , \rightarrow) \) is a commutative monoid,
3. \( z \leq (x \rightarrow y) \) iff \( x \ast z \leq y \) for all \( x, y, z \) (residuation),
4. \( (x \rightarrow y) \lor (y \rightarrow x) = 1 \) (prelinearity).

If the lattice reduct of \( B \) is linearly ordered we say that \( B \) is linear (or that \( B \) is an MTL-chain).

\(^7\)Note that in the older paper [31] the terms ‘Prelinear property’ and ‘fuzzy logics’ were used instead of ‘Semilinear property’ and ‘semilinear logics’.

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The first three conditions say that an MTL-algebra is a bounded commutative integral residuated lattice, see [52]. It is well known that the class of MTL-algebras forms a variety and the prelinearity guarantees the fact that this variety is generated by the class of MTL-chains. BL-algebras are MTL-algebras fulfilling one additional identity: $x \& (x \to y) = x \land y$ (see [66]).

**Definition 1.8 (Esteva and Godo [42])**. An MTL$\Delta$-algebra is a structure $B = (B, \lor, \land, \& \to, 0, 1)$ such that:

1. $(B, \lor, \land, \& \to, 0, 1)$ is an MTL-algebra,
2. $\Delta x \lor (\Delta x \to 0) = 1$,
3. $\Delta (x \lor y) \leq (\Delta x \lor \Delta y)$,
4. $\Delta x \leq x$,
5. $\Delta x \leq \Delta y$,
6. $\Delta (x \to y) \leq \Delta x \to \Delta y$,
7. $\Delta 1 = 1$.

The basic semantical notions of evaluation, tautology, and model are defined as usual. In the following definition by the term *additional connective (axiom)* of $L$ we understand a connective (axiom) not present in MTL.

**Definition 1.9**. Let $L$ be a core fuzzy logic and $I$ the set of additional basic connectives of $L$. An $L$-algebra is a structure $B = (B, \lor, \land, \& \to, 0, 1)$ such that $(B, \lor, \land, \& \to, 0, 1)$ is an MTL-algebra and each additional axiom of $L$ is a tautology of $B$.

We define the notion of an $L$-algebra for $\Delta$-core fuzzy logics analogously. The following proposition collects the basic properties of $(\Delta)$-core fuzzy logics which are either easy observations or consequences of the corresponding papers (see [37] for more details).

**Proposition 1.10**. Let $L$ be a $(\Delta)$-core fuzzy logic.

- $L$ is an implicational logic in the sense of Rasiowa [127].
- $L$ is a weakly implicational semilinear logic in the sense of Cintula and Noguera [37].
- $L$ is algebraizable in the sense of Blok and Pigozzi [25] with translation sets $\Delta(p, q) = \{p \equiv q\}$ and $E(p) = \{p = 1\}$.
- The class of $L$-algebras is a variety.
- The class of $L$-algebras is an equivalent algebraic semantics of $L$.
- Every $L$-algebra is representable as a subdirect product of $L$-chains.

The upcoming variant of the completeness theorem for $(\Delta)$-core fuzzy logics can be either proven analogously as the completeness of MTL in [42] or obtained as a corollary of results in [31].

**Theorem 1.11 (Strong Completeness Theorem)**. Let $L$ be a $(\Delta)$-core fuzzy logic, $\varphi$ a formula, and $T$ a theory. Then the following conditions are equivalent:

- $T \vdash \varphi$.
- $e(\varphi) = 1$ for each $L$-algebra $B$ and each $B$-model $e$ of theory $T$.
- $e(\varphi) = 1$ for each $L$-chain $B$ and each $B$-model $e$ of theory $T$.

**1.2. t-Norm based fuzzy logics**

As we mentioned above in fuzzy logic two classes of algebras play an important role: the linear ones and the so-called standard ones. Which algebras are standard? We claim that the answer to this question cannot be fully formal, as there is always a stipulating aspect—we are choosing ‘intended’ semantics for some logics and vice versa: we introduce logics to match our ‘intended’ semantics. We start by recalling the definition of a t-norm:

**Definition 1.12**. A t-norm is a commutative associative non-decreasing (in both arguments) binary operation $*$ on $[0, 1]$ with unit element 1 and absorbing element 0. A t-norm is (left-)continuous if it is a (left-)continuous mapping in the usual sense.
There are three special continuous t-norms. These t-norms have many important properties and each continuous t-norm is 'constructed' as their ordered sum (Mostert–Shields theorem, see e.g., [66] for details).

- **Łukasiewicz** t-norm: \( x \ast y = \max(0, x + y - 1) \).
- **Gödel** t-norm: \( x \ast y = \min(x, y) \).
- **Product** t-norm: \( x \ast y = x \cdot y \) (product of reals).

Left-continuity of a t-norm is necessary and sufficient for the existence of its residuum:

**Definition 1.13.** Let \( \ast \) be a left-continuous t-norm. Then the residuum \( \Rightarrow \ast \) of the t-norm \( \ast \) is defined as \( x \Rightarrow \ast y = \max\{z \mid x \ast z \leq y\} \). We say that \( ([0, 1], \ast, \Rightarrow \ast, \text{min}, \text{max}, 0, 1) \) is a t-algebra and denote it \( [0, 1]_\ast \).

Obviously each t-algebra \( [0, 1]_\ast \) is an MTL-chain. It can be shown that \( [0, 1]_\ast \) is a BL-chain iff \( \ast \) is a continuous t-norm. Now we expand the notion of t-algebra to the richer languages. We need to make the mentioned stipulations (recall that we are defining 'intended' semantics).

**Definition 1.14.** Let \( I \subseteq \{\Delta, \sim\} \cup \{\delta_n \mid n \in \mathbb{N}\} \cup \{\overline{r} \mid r \in [0, 1]\} \). We say that an algebra \( B = ([0, 1], \ast, \Rightarrow \ast, \text{min}, \text{max}, (c_B)_{c \in I}, 0, 1) \) is a t-algebra iff \( ([0, 1], \ast, \Rightarrow \ast, \text{min}, \text{max}, 0, 1) \) is a t-algebra and the following holds:

<table>
<thead>
<tr>
<th>if ( c \in I ) is</th>
<th>Then ( c_B ) is</th>
<th>It was first used in</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \Delta )</td>
<td>( \Delta_B(1) = 1, \Delta_B(x) = 0 ) otherwise</td>
<td>[2]</td>
</tr>
<tr>
<td>( \sim )</td>
<td>( \sim_B(x) = 1 - x )</td>
<td>[43]</td>
</tr>
<tr>
<td>( \overline{r} )</td>
<td>( \overline{r}_B = r )</td>
<td>[66]</td>
</tr>
<tr>
<td>( \delta_n )</td>
<td>( (\delta_n)_B(x) = \frac{x}{n} )</td>
<td>[54]</td>
</tr>
</tbody>
</table>

Of course, this is not a complete definition (however, it contains the majority of known cases from the fuzzy logic literature), thus if we introduce a new connective and we want to expand the notion of t-algebra to cover also this case we need to specify the standard semantics of this connective.

**Definition 1.15.** Let \( L \) be a \( \Delta \)-core fuzzy logic. By \( T_L \) we denote the set of \( L \)-algebras which are t-algebras. These algebras are called standard \( L \)-algebras.

In particular, the set of standard BL-algebras consists of all algebras with domain \([0, 1]\). The set \( T_L \) could be empty, for example classical logic is a core fuzzy logic and has no standard algebra (in the sense of the previous definition). The notion of standard semantic is sometimes even further restricted. Because in several prominent fuzzy logics (e.g., Łukasiewicz and Product) all standard algebras are mutually isomorphic (in Gödel logic there is only one standard algebra) we choose a particular representative and call it THE standard algebra of the logic in question, e.g., in Łukasiewicz, product, and Gödel logics we chose the t-algebras given by the corresponding t-norms.

**Definition 1.16.** A \( \Delta \)-core fuzzy logic \( L \) is called t-norm based (or t-fuzzy) logic if it is complete w.r.t. the class of standard \( L \)-algebras, i.e., if for each formula \( \varphi \) the following conditions are equivalent:

- \( \vdash \varphi \).
- \( \varphi \) is a \( B \)-tautology for every standard \( L \)-algebra \( B \).

We say that t-norm based fuzzy logics enjoy standard completeness. Sometimes we can do better:

**Definition 1.17.** Let \( L \) be a t-fuzzy logic. We say that \( L \) enjoys (finite) strong standard completeness if for each (finite) theory \( T \) and each formula \( \varphi \) the following conditions are equivalent:

- \( T \vdash \varphi \).
- \( e(\varphi) = 1 \) for each standard \( L \)-algebra \( B \) and each \( B \)-model \( e \) of \( T \).
There is a recent paper [32] studying completeness properties of fuzzy logic w.r.t. different distinguished semantics (finite-, rational-, real-, hyperreal-valued). The real-valued semantics is close to our standard one, but real-valued algebras just need to have their lattice reducts being the interval $[0,1]$ with the usual order of reals and are NOT required to fulfill the additional conditions of Definition 1.14. However, real-valued semantic is also referred to as ‘standard’ in that paper (in fact there is an ongoing debate in the fuzzy logic community WHAT the standard semantics is) and our ‘standard’ one is there called ‘canonical’ (but only in the case of constants $\langle \rangle$). We hope that this explanation helps to avoid a possible confusion.

The majority of fuzzy logics listed in the beginning of Section 1 are t-fuzzy logics, some of them enjoy (finite) strong standard completeness:

Example 1.18.

- The following t-fuzzy logics have strong standard completeness: MTL, IMTL, NM, WNM, Gödel logic, expansions of logics mentioned till now by $\triangle$, $G_{\sim}$ (See [100,40,86,66,43]).
- The following t-fuzzy logics have finite strong standard completeness but not strong standard completeness: BL, $\Pi$MTL, Łukasiewicz logic, product logic, RPL, expansions of logics mentioned till now (in this example) by $\triangle$, SBL, SBL$\triangle$, SBL$\sim$, $\Pi I_1$, and $\Pi I_2$. (See [30,95,66,86,43,97,44]).
- Some t-fuzzy logics have not even finite strong standard completeness: e.g., product logic with rational constants (see [41]).
- The following $(\triangle)$-core fuzzy logics are not t-norm based: $\Pi \sim$, $PL$, any proper extension of Łukasiewicz logic (including finite valued ones and classical logic), etc. (see [43,97]).

In the usual practice we have some intended semantics (e.g., Łukasiewicz t-norm) and we want to define the logic ‘corresponding’ to this semantics:

**Theorem 1.19.** Let $T$ be a set of t-algebras in a fixed language. Let $L_T$ be the logic axiomatized by all $T$-tautologies and modus ponens. Then $L_T$ is a t-fuzzy logic.

For a left-continuous t-norm $*$ we write $L_*$ instead of $L_{[0,1]}$. At the end of the section, we give the universal-algebraic description of the previously defined notions:

**Theorem 1.20** (Cintula et al. [32]). Let $L$ be a $(\triangle)$-core fuzzy logic. Then the following are equivalent

- $L$ is t-norm based fuzzy logic.
- The variety of $L$-algebras is generated by $T_L$.

Furthermore, we can show that the following are equivalent:

- $L$ has finite strong standard completeness.
- The variety of $L$-algebras is generated by $T_L$ as a quasivariety.
- Each non-trivial $L$-algebra can be partially embedded into some algebra from $T_L$.

Finally, we can show that the following are equivalent:

- $L$ has strong standard completeness.
- Each at most countable non-trivial $L$-chain can be embedded into some algebra from $T_L$.

2. $(\triangle)$-core predicate fuzzy logics

2.1. Syntax

In the following let $L$ be a fixed $(\triangle)$-core fuzzy logic in a propositional language $\mathcal{L}$. We define the corresponding predicate fuzzy logic $L\forall$. Notice that unlike in [66] we put no restriction on the cardinality of predicate languages.

**Definition 2.1** (Predicate language). A predicate language $\Gamma$ is a triple $(P, F, A)$. $P$ is a non-empty set of predicate symbols, $F$ is a set of function symbols, and $A$ is a function assigning to each predicate and function symbol a natural
number called the \textit{arity of the symbol}. The functions \( f \) for which \( A(f) = 0 \) are called the \textit{object constants}. The predicates \( P \) for which \( A(P) = 0 \) are called the \textit{truth constants}.

Let us denote the set of object constants by \( C \). In the following let \( \Gamma \) be a fixed predicate language for the logic \( L \mathcal{V} \). The following definitions are absolutely standard, but we present them for the reader’s convenience.

**Definition 2.2 (Terms).** Each object variable is a \( \Gamma \)-\textit{term}. Let \( f \) be a function symbol, \( t_1, \ldots, t_n \) be \( \Gamma \)-\textit{terms}, and \( A(f) = n \). Then \( f(t_1, \ldots, t_n) \) is a \( \Gamma \)-\textit{term}. All \( \Gamma \)-\textit{terms} are constructed in this way.

**Definition 2.3 (Formulas).** Let \( t_1, \ldots, t_n \) be \( \Gamma \)-\textit{terms}, \( P \) a predicate symbol, and \( A(P) = n \). Then \( P(t_1, \ldots, t_n) \) is an \textit{atomic} \( \Gamma \)-\textit{formula}. The nullary logical connectives of \( L \) and truth constants are atomic \( \Gamma \)-\textit{formulas} as well. All \( \Gamma \)-\textit{formulas} are constructed in this way.

Let \( \varphi \) be a \( \Gamma \)-\textit{formula} and \( x \) an object variable. Then \( (\forall x)\varphi \) and \( (\exists x)\varphi \) are \( \Gamma \)-\textit{formulas}. Furthermore, the class of \( \Gamma \)-\textit{formulas} is closed under logical connectives of \( L \). All \( \Gamma \)-\textit{formulas} are constructed in this way.

Notice that the set of terms depends on \( \Gamma \) only, whereas the set of formulas depends on the propositional language as well. So we should speak about \( \Gamma \)-\textit{terms} and \( (L, \Gamma) \)-\textit{formulas}. However, we speak about \( \Gamma \)-\textit{formulas} if the propositional language is clear from the context and we speak about terms and formulas if both propositional and the predicate languages are clear from the context.

**Definition 2.4 (Bound and free occurrence).** An occurrence of a variable \( x \) in a formula \( \varphi \) is bound if it is in the scope of a quantifier over \( x \). A formula \( \varphi \) is said to be \textit{closed (open)} if there are no free (bound) occurrences of variables in \( \varphi \).

Closed formulas are also known as \textit{sentences}. Unless stated otherwise, by \( \varphi(x_1, \ldots, x_n) \) we mean that all free variables of \( \varphi \) are among \( x_1, \ldots, x_n \).

**Definition 2.5 (Theory).** A set of \( \Gamma \)-\textit{sentences} is called a \( \Gamma \)-\textit{theory}.

Instead of \( \bar{c}_1, \ldots, \bar{c}_n, (n \text{ is arbitrary or fixed by the context}) \) we shall sometimes write just \( \bar{c} \). If \( \varphi(x_1, \ldots, x_n, \bar{z}) \) is a formula and we substitute terms \( t_i \) for all \( x_i \)’s in \( \varphi \), we denote the resulting formula in the context simply by \( \varphi(t_1, \ldots, t_n, \bar{z}) \).

**Definition 2.6 (Substitutability).** A term \( t \) is substitutable for the object variable \( x \) in a formula \( \varphi(x, \bar{z}) \) if no occurrence of any variable occurring in \( t \) is bound in \( \varphi(t, \bar{z}) \) unless it was already bound in \( \varphi(x, \bar{z}) \).

Now we introduce an axiomatic system for the predicate fuzzy logic \( L \mathcal{V} \), later (in Section 3) we modify this system in several ways.

**Definition 2.7 (Predicate logic).** Let \( L \) be a \( (\Delta-)\text{core fuzzy logic}. The logic \( L \mathcal{V} \) has axioms:

(P) the axioms resulting from the axioms of \( L \) by the substitution of the propositional variables by the \( \Gamma \)-\textit{formulas},

(\forall 1) \( (\forall x)\varphi(x) \rightarrow \varphi(t) \), where \( t \) is substitutable for \( x \) in \( \varphi \),

(\exists 1) \( \varphi(t) \rightarrow (\exists x)\varphi(x) \), where \( t \) is substitutable for \( x \) in \( \varphi \),

(\forall 2) \( (\forall x)(\chi \rightarrow \varphi) \rightarrow (\chi \rightarrow (\forall x)\varphi) \), where \( x \) is not free in \( \chi \),

(\exists 2) \( (\forall x)(\varphi \rightarrow \chi) \rightarrow ((\exists x)\varphi \rightarrow \chi) \), where \( x \) is not free in \( \chi \),

(\forall 3) \( (\forall x)(\chi \vee \varphi) \rightarrow (\forall x)\varphi \), where \( x \) is not free in \( \chi \).

The deduction rules are those of \( L \) and \textit{generalization}: from \( \varphi \) infer \( (\forall x)\varphi \).

Before we proceed we show that the transition to predicate logic does not alter the deduction theorems and both properties PCP and SLP.

**Theorem 2.8 (Hájek and Cintula [84]).** Let \( L \) be a core fuzzy logic. Then \( L \mathcal{V} \) has Local Deduction Theorem (\( LDT \)): for each theory \( T \), and sentences \( \varphi, \psi \) holds: \( T, \varphi \vdash \psi \) iff there is natural \( n \) such that \( T \vdash \varphi^n \rightarrow \psi \).
Let \( L \) be a \( \Delta \)-core fuzzy logic. Then \( \forall \) has Delta Deduction Theorem (\( DT_\Delta \)): for each theory \( T \), and sentences \( \varphi, \psi \) holds: \( T \vdash \Delta \varphi \rightarrow \psi \).

**Corollary 2.9.** Let \( L \) be \((\Delta-)\) core fuzzy logic, \( T \) a theory, and \( \varphi, \psi, \chi \) sentences. Then the following holds in \( L \forall \):

1. **Proof by cases property:** if \( T \), \( \varphi \vdash \chi \) and \( T, \psi \vdash \chi \), then \( T \vdash \varphi \lor \psi \chi \).
2. **Semilinearity property:** if \( T \), \( \varphi \vdash \psi \chi \) and \( T, \psi \rightarrow \varphi \chi \), then \( T \vdash \varphi \chi \).

The following corollary is [66, Theorem 5.2.15] for BL. In [84] we give a general proof-theoretic proof. \( \mathcal{LDT} \) is essential in the proof and the analogous statement for \( \Delta \)-core fuzzy logics does not hold, because \( \Delta(\exists x)\varphi \rightarrow (\exists x)\Delta \varphi \) is not a theorem. This is the reason why some of the upcoming results will be proven for core fuzzy logics only.

**Corollary 2.10.** Let \( \Gamma \) be a predicate language, \( L \) a core fuzzy logic, \( T \) a \( \Gamma \)-theory, \( c \) a constant not appearing in \( \Gamma \), and \( \varphi(x) \) a \( \Gamma \)-formula. Then \( T \cup \{\varphi(c)\} \) is a conservative extension of \( T \cup \{\exists x)\varphi(x)\} \).

2.2. **Semantics**

To simplify the formulation of upcoming definitions we fix: a \((\Delta-)\)core fuzzy logic \( L \) in a propositional language \( \mathcal{L} \), predicate language \( \Gamma \), and an \( \mathcal{L} \)-algebra \( B \).

**Definition 2.11 (Structure).** A \( B \)-structure \( M \) for \( \Gamma \) has the form: \( M = (M, (P_M)_{P \in \mathcal{P}}, (f_M)_{f \in \mathcal{F}}) \), where \( M \) is a non-empty domain; \( P_M \) is an \( n \)-ary fuzzy relation \( M^n \rightarrow B \) for each \( n \)-ary predicate symbol \( P \in \mathcal{P} \) with \( n \geq 1 \) and an element of \( B \) if \( P \) is a truth constant; \( f_M \) is a function \( M^n \rightarrow M \) for each \( n \)-ary function symbol \( f \in \mathcal{F} \) with \( n \geq 1 \) and an element of \( M \) if \( f \) is an object constant.

**Definition 2.12 (Evaluation).** Let \( M \) be a \( B \)-structure for \( \Gamma \). An \( M \)-evaluation of the object variables is a mapping \( v \) which assigns to each variable an element from \( M \).

Let \( v \) be an \( M \)-evaluation, \( x \) a variable, and \( a \in M \). Then \( v(x \rightarrow a) \) is an \( M \)-evaluation such that \( v(x \rightarrow a)(x) = a \) and \( v(x \rightarrow a)(y) = v(y) \) for each object variable \( y \) different from \( x \).

**Definition 2.13 (Truth definition).** Let \( M \) be a \( B \)-structure for \( \Gamma \) and \( v \) an \( M \)-evaluation. We define values of the terms and truth values of the formulas in \( M \) for an evaluation \( v \) as

\[
\|x\|_{M,v} = v(x),
\]

\[
\|f(t_1, \ldots, t_n)\|_{M,v} = f_M(\|t_1\|_{M,v}, \ldots, \|t_n\|_{M,v}) \quad \text{for } f \in \mathcal{F}
\]

\[
\|P(t_1, \ldots, t_n)\|_{M,v} = P_M(\|t_1\|_{M,v}, \ldots, \|t_n\|_{M,v}) \quad \text{for } P \in \mathcal{P}
\]

\[
\|c(\varphi_1, \ldots, \varphi_n)\|_{M,v} = c_B(\|\varphi_1\|_{M,v}, \ldots, \|\varphi_n\|_{M,v}) \quad \text{for } c \in \mathcal{L}
\]

\[
\|(\forall x)\varphi\|_{M,v} = \inf\{\|\varphi\|_{M,v[x \rightarrow a]} | a \in M\}.
\]

\[
\|(\exists x)\varphi\|_{M,v} = \sup\{\|\varphi\|_{M,v[x \rightarrow a]} | a \in M\}.
\]

If the infimum or supremum does not exist, we take its value as undefined. We say that \( M \) is safe iff \( \|\varphi\|_{M,v} \) is defined for each \( \Gamma \)-formula \( \varphi \) and each \( M \)-evaluation \( v \).

We set the following useful denotations. We write:

- \( \|\varphi(a_1, \ldots, a_n)\|_{(B,M)} \) instead of \( \|\varphi(x_1, \ldots, x_n)\|_{(B,M,v)} \) for \( v(x_i) = a_i \).
- \( (B,M) \models \varphi[v] \) if \( \|\varphi\|_{M,v} = 1 \).
- \( (B,M) \not\models \varphi \) if \( (B,M) \not\models \varphi[v] \) for each \( M \)-evaluation \( v \).
- \( B \models \varphi \) if \( (B,M) \models \varphi \) for each safe \( B \)-structure \( M \) (we also say that \( \varphi \) is a \( B \)-tautology).

When \( B \) is known from the context we write \( M \models \varphi \) only.
1 **Definition 2.14 (Model).** Let $M$ be a safe $B$-structure for $\Gamma$ and $T$ a $\Gamma$-theory. $M$ is called a $B$-model of $T$ if $(B, M) \vDash \varphi$ for each $\varphi \in T$.

Observe that models are safe structures (by the definition). As obviously each safe $B$-structure is a $B$-model of the empty theory, we use the term model only in the rest of the text. We say that $(B, M)$ is linearly ordered. Analogously we use term $s$-model for those over standard algebras. Thus if we say ‘for each ($\ell$-/s-)model $(B, M)$ of $T$’ we mean ‘for each (linear/standard) $L$-algebra $B$ and each safe $B$-model $M$ of $T$’.

2 **Definition 2.15 (Exhaustive model).** Let $(B, M)$ be a model. By $\text{Alg}(B, M)$ we denote the subalgebra of $B$ whose domain is the set $\{\|\varphi\|^B_{M, v} | \varphi \text{ a formula and } v \text{ an } M\text{-evaluation}\}$.

We could see that $(B, M)$ is exhaustive if $B$ does not contain any ‘unnecessary’ elements. Clearly, for each $\varphi$ and $v$,

$\|\varphi\|^B_{M, v} = \|\varphi\|^\text{Alg}(M, B)_{M, v}$. The notion of an exhaustive model first appeared in [74].

3. **Fundamental lemma and the completeness theorem**

We start by several important definitions. Obviously, all of them depend on the logic in question; however, to simplify the terminology we assume that the logic is always known from the context. We survey results of [84] generalizing those of [66].

**Definition 2.16 (Linear theory).** A theory $T$ is linear if for each pair $\varphi, \psi$ of sentences we have $T \vdash \varphi \rightarrow \psi$ or $T \vdash \psi \rightarrow \varphi$.

In the existing literature linear theories are often called complete theories.

**Definition 2.17 (Henkin and $\exists$-Henkin theories).** Let $\Gamma, \Gamma'$ be predicate languages such that $\Gamma \subseteq \Gamma'$, and $T$ a $\Gamma'$-theory. We say that $T$ is $\Gamma$-Henkin if for each $\Gamma$-sentence $\varphi = (\forall x)\psi$ such that $T \vdash \varphi$ there is a constant $c$ in $\Gamma'$ such that $T \vdash \psi(c)$.

A theory is called $\exists$-$\Gamma$-Henkin if for each $\Gamma$-sentence $(\exists y)\psi(y)$ it holds: if $T \vdash (\exists y)\psi(y)$, then there is a constant $c$ in $\Gamma'$ such that $T \vdash \psi(c)$.

Finally, a theory is called doubly-$\Gamma$-Henkin if it is both $\exists$-$\Gamma$-Henkin and $\Gamma$-Henkin. If $\Gamma = \Gamma'$ we say that $T$ is Henkin ($\exists$-Henkin, doubly Henkin).

Assume that we have a theory $T$ and a formula $\varphi$, such that $T \not\vdash \varphi$. We want to keep $\varphi$ unprovable in any consistent extension of $T$. In classical logic, we just add $\neg \varphi$ to $T$ and we are done. In fuzzy logics the situation is not that simple and so we need to ‘store’ the formulas we want to keep unprovable in a special theory $\Psi$. First, we introduce a concept of directed theory.

**Definition 2.18 (Directed theory).** A theory $\Psi$ is directed if for each $\varphi, \psi \in \Psi$ there is $\chi \in \Psi$ such that both $\varphi \rightarrow \chi$ and $\psi \rightarrow \chi$ are provable (we call $\chi$ an upper bound of $\varphi$ and $\psi$).

If $T$ and $\Psi$ are theories, by $T \not\vdash \Psi$ we mean $T \not\vdash \psi$ for each $\psi \in \Psi$. Now we can formulate the fundamental lemma:

**Lemma 2.19 (Fundamental lemma).** Let $L$ be a ($\Delta$)-core fuzzy logic, $\Gamma$ a predicate language, and $T, \Psi$ theories such that $T \not\vdash \Psi$ and $\Psi$ is directed.

1. There are $\Gamma$-theories $T', \Psi'$ such that $\Gamma \subseteq \Gamma'$, $T \subseteq T'$, $\Psi \subseteq \Psi'$, $T' \not\vdash \Psi'$, and for each theory $S$ (in arbitrary predicate language) it holds: if $T' \subseteq S$ and $S \not\vdash \Psi'$, then $S$ is $\Gamma$-Henkin.

2. If $L$ is a core fuzzy logic, then there is a $\Gamma$'-theory $T'$ such that $\Gamma \subseteq \Gamma'$, $T \subseteq T'$, $T' \not\vdash \Psi$, and for each theory $S$ (in arbitrary predicate language) it holds: if $T' \subseteq S$ and $S \not\vdash \Psi$, then $S$ is $\exists$-$\Gamma$-Henkin.

3. There is a linear $\Gamma$'-theory $T'$ such that $T \subseteq T'$ and $T' \not\vdash \Psi$.

Now we formulate two theorems (one for core and one for $\Delta$-core fuzzy logics) analogous to [66, Lemma 5.2.7].

These theorems work for a wider class of logics, predicate languages of arbitrary cardinality, and for keeping unprovable...
Let directed the completeness theorem for the (△Theorem 2.25 (CM

\&fi9850•T

Theorem 2.25. Let L be a (△-core fuzzy logic, Γ a predicate language, and T, Ψ theories such that T\(\not\vdash\)Ψ and Ψ is directed. Then there is a linear doubly Henkin Γ′-theory T ′ such that Γ ⊆ Γ′, T ⊆ T ′, and T ′\(\not\vdash\)Ψ.

Theorem 2.21. Let L be a (△-core fuzzy logic, Γ a predicate language, and T, Ψ theories such that T\(\not\vdash\)Ψ and Ψ is directed. Then there is a linear Henkin Γ′-theory T ′ such that Γ ⊆ Γ′, T ⊆ T ′ and T ′\(\not\vdash\)Ψ.

Now we define the notion of Lindenbaum algebra of a theory T and of the canonical model of a theory T.

Definition 2.22. Let L be a logic and T be a Γ-theory. We define \([\varphi]_T = \{\psi | T\vdash \varphi \iff \psi\}\) and \(L_T = \{[\varphi]_T | \varphi \text{ a closed formula}\}\). The Lindenbaum algebra of the theory T (denoted as \(\text{Lind}_T\)) is the algebra with the domain \(L_T\) and operations \(\sigma_{\text{Lind}_T}([\varphi_1]_T, \ldots, [\varphi_n]_T) = [\sigma(\varphi_1, \ldots, \varphi_n)]_T\) (for each connective \(\sigma\) of the logic L).

Definition 2.23. Let T be a linear Henkin Γ-theory. Then we define the canonical model of theory T, denoted by \((\text{Lind}_T, \text{CM}(T))\), where \(\text{Lind}_T\) is the Lindenbaum algebra of theory T, the domain of \(\text{CM}(T)\) consists of the closed Γ-terms, \(f_{\text{CM}(T)}(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)\) for each n-ary function symbol f, and \(P_{\text{CM}(T)}(t_1, \ldots, t_n) = [P(t_1, \ldots, t_n)]_T\) for each n-ary predicate symbol P.

We sometimes write only \(\text{CM}(T)\) instead of \((\text{Lind}_T, \text{CM}(T))\).

Lemma 2.24. Let L be a (△-)core fuzzy logic and T a Henkin theory. Then

- \(\text{Lind}_T\) is an L-chain iff T is linear.
- \([\varphi]_{\text{CM}(T)} = [\varphi]_T\).
- \(T\vdash \varphi \iff \text{CM}(T)\models \varphi\).
- \(\text{CM}(T)\) is exhaustive.

This lemma demonstrates the soundness of Definition 2.23 (\(\text{CM}(T)\) is indeed a model of T). Finally, we formulate the completeness theorem for the (△-)core predicate fuzzy logics.

Theorem 2.25 (Strong Completeness Theorem). Let L be (△-)core fuzzy logic, Γ a predicate language, T a theory, and \(\varphi\) a formula. Then the following are equivalent:

- \(T\vdash \varphi\).
- \((B, M)\models \varphi\) for each \(\ell\)-model \((B, M)\) of the theory T.
- \((B, M)\models \varphi\) for each exhaustive \(\ell\)-model \((B, M)\) of the theory T.

This theorem gives us the strong completeness of (△-)core predicate fuzzy logics (and in particular of all those mentioned in the beginning of Section 1) w.r.t. the class of its linearly ordered algebras. It also demonstrates that we can restrict ourselves to the exhaustive models. In Section 3.3 we will see that in some cases we can restrict ourselves even further: to the class of the so-called witnessed models.

Recall that many instances of this theorem were proven for particular (△-)core fuzzy logics independently: e.g., for Łukasiewicz logic in [20], for BL (and its axiomatic extensions) in [66], for MTL (and its axiomatic extensions) in [42], etc.

2.4. Some particular t-norm based predicate fuzzy logics

In this subsection we recall important properties of predicate fuzzy logics (standard completeness, \((\forall 3)\)-eliminability, and \(3\)-definability) and survey which particular predicate t-fuzzy logics enjoy them. We shall assume that the reader
Table 3
Properties of important predicate t-fuzzy logics.

<table>
<thead>
<tr>
<th>Logic</th>
<th>SC</th>
<th>FSSC</th>
<th>SSC</th>
<th>∀-3-elim.</th>
<th>∃-defin.</th>
</tr>
</thead>
<tbody>
<tr>
<td>IMTL, NM</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes [40,42]</td>
<td>No^a</td>
<td>Yes</td>
</tr>
<tr>
<td>MTL, SMTL, WNM, G</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes [113,40,42,66]</td>
<td>No [86]</td>
<td>No</td>
</tr>
<tr>
<td>L</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
</tr>
<tr>
<td>IMTL, II</td>
<td>?</td>
<td>No</td>
<td>[112]</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>BL, SBL, II</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>

^a As far as we know this is a new result: consider a rotation [99,53] of the algebra used in [86, Example 1] do demonstrate failure of ∀-3-elimination in Gödel logic. The resulting algebra is clearly an NM-algebra and the counterexample runs in the same way.

Table 4
Properties of important predicate t-fuzzy logics with △.

<table>
<thead>
<tr>
<th>Logic</th>
<th>SC</th>
<th>FSSC</th>
<th>SSC</th>
<th>∀-3-elim.</th>
<th>∃-defin.</th>
</tr>
</thead>
<tbody>
<tr>
<td>IMTL△, NM△</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>MTL△, SMTL△, G△, WNM△</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
</tr>
<tr>
<td>G△</td>
<td>Yes</td>
<td>Yes</td>
<td>Yes</td>
<td>?</td>
<td>Yes</td>
</tr>
<tr>
<td>SBL</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>?</td>
<td>Yes</td>
</tr>
<tr>
<td>L△, PL△, LII</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
<td>Yes</td>
</tr>
<tr>
<td>BL△, SBL△, II△, IMTL△</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>

Table 5
Theorems of important predicate t-fuzzy logics.

<table>
<thead>
<tr>
<th>Logic</th>
<th>(C∃)</th>
<th>(CY)</th>
<th>(C∃△)</th>
<th>(CY△)</th>
<th>(T)</th>
</tr>
</thead>
<tbody>
<tr>
<td>L</td>
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<td>Yes</td>
<td>–</td>
<td>–</td>
<td>Yes</td>
</tr>
<tr>
<td>II</td>
<td>Yes</td>
<td>No</td>
<td>–</td>
<td>–</td>
<td>Yes</td>
</tr>
<tr>
<td>G</td>
<td>No</td>
<td>No</td>
<td>–</td>
<td>–</td>
<td>Yes</td>
</tr>
<tr>
<td>BL, SBL</td>
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<td>No</td>
<td>–</td>
<td>–</td>
<td>No</td>
</tr>
<tr>
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<td>Yes</td>
<td>Yes</td>
<td>No</td>
<td>No</td>
<td>Yes</td>
</tr>
<tr>
<td>G△</td>
<td>No</td>
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<td>No</td>
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</tr>
<tr>
<td>BL△, SBL△</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
<td>No</td>
</tr>
</tbody>
</table>

1 knows the corresponding particular propositional t-fuzzy logics (see the survey [64]). We summarize known results in Tables 3–5. The results concerning arithmetical complexity of the sets of standard tautologies (and other important sets of formulas) of particular t-fuzzy logics is summarized in Section 4.

Before we turn our attention to the t-norm based fuzzy logic we list several theorems which hold in MTL∀ and thus they hold in all (∆-)core fuzzy logics.

Theorem 2.26. Assume that v does not contain x freely. The following are theorems of each (∆-)core predicate fuzzy logic:

(T∀1) (∀x)(v → φ) ≡ (v → (∀x)φ).
(T∀2) (∀x)(φ → v) ≡ ((∃x)φ → v).
(T∀3) (∃x)(v → φ) → (v → (∃x)φ).
(T∀4) (∃x)(φ → v) → ((∀x)φ → v).
(T∀5) (∀x)(φ → ψ) → ((∀x)φ → (∀x)ψ).
(T∀6) (∀x)(φ → ψ) → ((∃x)φ → (∃x)ψ).
(T∀7) (∀x)φ& (∃x)ψ → (∃x)(φ&ψ).

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Let \( L \) be a \((\triangle-)\)core predicate fuzzy logic. We say \( L \) enjoys \((\forall 3)\)-eliminability if \( L \forall \) is redundant.

Later we will see that \((\forall 3)\)-eliminability means that logics \( L \forall \) and \( L \forall^b \) coincide (see Section 3.1).

\textbf{Definition 2.30} \((\exists\text{-}Definability)\). Let \( L \) be a \((\triangle-)\)core predicate fuzzy logic. We say that \( L \) has \( \exists\text{-}definability \) if there is a unary connective \( \sim \) in the language of \( L \) such that \( L \forall^{\exists}(\exists x)\varphi \equiv \sim(\forall x)\sim\varphi \).

\[\begin{align*}
& (T\forall 8) \ (\forall x)\varphi(x) \equiv (\forall y)\varphi(y). \\
& (T\forall 8') \ (\exists x)\varphi(x) \equiv (\exists y)\varphi(y). \\
& (T\forall 9) \ ((\varphi \& \psi) \equiv ((\exists x)\varphi \& (\exists x)\psi). \\
& (T\forall 10) \ (\exists x)\varphi^n \equiv ((\exists x)\varphi)^n \text{ for each } n \geq 1. \\
& (T\forall 11) \ (\exists x)\varphi \rightarrow \neg(\forall y)\neg\varphi. \\
& (T\forall 12) \ \neg(\exists x)\varphi \equiv (\forall y)\neg\varphi. \\
& (T\forall 13) \ (\exists x)(\varphi \land \psi) \equiv (\varphi \land (\exists x)\psi). \\
& (T\forall 14) \ (\exists x)(\varphi \lor \psi) \equiv (\varphi \lor (\exists x)\psi). \\
& (T\forall 15) \ (\forall x)(\varphi \land \psi) \equiv (\varphi \land (\forall x)\psi). \\
& (T\forall 16) \ (\exists x)(\varphi \lor \psi) \equiv ((\exists x)\varphi \lor (\exists x)\psi). \\
& (T\forall 17) \ (\forall x)(\varphi \land \psi) \equiv ((\forall x)\varphi \land (\forall x)\psi).
\end{align*}\]

Recall that \((B, M)\) is an \( s \)-model (of \( T \)) iff \( B \) is a standard \( L \)-algebra.

\textbf{Definition 2.27} (Standard completeness). Let \( L \) be a t-fuzzy logic. We say that \( L \forall \) enjoys (finite) standard completeness, \((F)SSC\) in short, if for each (finite) theory \( T \) and each formula are the following conditions equivalent:

- \( T \vdash \varphi \)
- \( (B, M) \models \varphi \) for each \( s \)-model \((B, M)\) of the theory \( T \).

We say that \( L \forall \) enjoys standard completeness \((SC)\) if the above equivalence holds for the empty theory \( T \).

Observe that this definition makes sense for t-fuzzy logics only (as we need to know that standard algebras are).

Obviously, strong standard completeness entails finite strong standard completeness and it entails standard completeness. However, unlike in the propositional case (see Example 1.18) we do not know any fuzzy logics which would demonstrate the strictness of the above implications (see Tables 3 and 4; see also paper [32] for completeness results w.r.t. other distinguished semantics). Another interesting observation is that each known propositional t-fuzzy logic enjoys standard completeness iff its predicate version enjoys standard completeness.

Unlike in the propositional case (see Theorem 1.20) we cannot present a simple algebraic characterization of the notion of strong standard completeness. One could be tempted to claim the equivalence of:

- \( L \forall \) has strong standard completeness.
- Each at most countable non-trivial \( L \)-chain can be embedded into some algebra from \( T_L \) and this embedding preserves all suprema and infima.

However, in [32], the authors show that the above statement does not hold in general for arbitrarily distinguished semantics (the counterexample involves hyperreal semantics and Łukasiewicz logic, and so it still could be true for standard semantics which is yet another interesting open problem). They also prove a model-theoretic variant of the above equivalence (for the needed model-theoretic notions see Section 5).

\textbf{Theorem 2.28}. Let \( L \) be a \((\triangle-)\)core fuzzy logic. Then the following are equivalent:

- \( L \forall \) has strong standard completeness.
- For every countable non-trivial \( L \)-chain \( A \) and every model \((A, M)\) there is an \( s \)-model \((B, M')\) such that \((A, M)\) can be elementarily embedded into \((B, M')\) (or is elementarily equivalent to) \((B, M')\).

The next two notions distinguish logics where the axiomatic systems can be simplified.

\textbf{Definition 2.29} \((\forall 3)-\text{Eliminability}\). Let \( L \) be a \((\triangle-)\)core predicate fuzzy logic. We say \( L \forall \) has \((\forall 3)\)-eliminability if axiom \((\forall 3)\) is redundant.

\begin{align*}
& (i) \ L \forall \text{ has strong standard completeness.} \\
& (ii) \text{ For every countable non-trivial } L \text{-chain } A \text{ and every model } (A, M) \text{ there is an } s \text{-model } (B, M') \text{ such that } (A, M) \\
& \text{ can be elementarily embedded into } (B, M') \text{ (or is elementarily equivalent to) } (B, M').
\end{align*}
Theorem 2.31. Let \( L \) be a \((\Delta\text{-})\)core logic and \( \sim \) a unary (definable) connective in the language of \( L \) such that
\[
\varphi \rightarrow \psi \vdash L \sim \psi \rightarrow \sim \varphi \text{ and } \vdash L \varphi \equiv \sim \sim \varphi \text{ (i.e., } \sim \text{ is an involutive negation). Then } L \forall \text{ has } \exists\text{-definability.}
\]

Theorem 2.32. Let \( L \forall \) have \( \exists\text{-definability. Then axioms (31) and (32) are redundant.} \]

Finally there are several important formulas which are (non-)provable (see Table 5) in particular predicate t-fuzzy logics. We give their list with their usual names:

Definition 2.33. We shall discuss the following axioms (assume that \( v \) does not contain \( x \) freely):

1. \((C \forall) (\exists y)((\exists x)\varphi(x) \rightarrow \varphi(y)). \]
2. \((C \forall) (\exists y)(\varphi(y) \rightarrow (\forall x)\varphi(x)). \]
3. \((C \forall) (\exists y)\Delta((\exists x)\varphi(x) \rightarrow \varphi(y)). \]
4. \((C \forall) (\forall x)\Delta(\varphi(y) \rightarrow (\forall x)\varphi(x)). \]
5. \((T) (\forall x)(\varphi \& v) \equiv ((\forall x)\varphi \& v). \]

Notice that clearly \((C \forall) \exists \) implies \((C \forall) \) and analogously for \((C \forall) \). We can simple prove that axioms \((C \forall) \) and \((C \forall) \) (which will play an important role in Section 3.3) are equivalent to the two missing quantifier shift axioms for implication (converse implications of \((T \forall 3)\) and \((T \forall 4)\)) and \((C \forall) \) entails the final missing quantifier shift for strong conjunction (i.e., axiom \((T)\)).

Lemma 2.34. MTL\( \forall \) proves:

- \((\exists y)((\exists x)\varphi(x) \rightarrow \varphi(y))(v \rightarrow (\exists x)\varphi(x) \rightarrow (\exists y)\varphi(y)). \]
- \((\exists y)(\varphi(y) \rightarrow (\forall x)\varphi(x))(\exists y)(\exists x)\varphi(x) \rightarrow \varphi(y)). \]
- \((\exists y)\Delta(\varphi(y) \rightarrow (\forall x)\varphi(x))(\exists y)(\exists x)\varphi(x) \rightarrow \varphi(y)). \]
- \((\exists y)(\varphi(y) \rightarrow (\forall x)\varphi(x))(\exists y)(\exists x)\varphi(x) \rightarrow \varphi(y)). \]

Theorem 2.35. Let \( L \) be a core fuzzy logic in the language of MTL, \( \Gamma \) a predicate language, and \( T \) a theory containing instances of axioms \((C \exists)\) and \((C \forall)\) for all \( \Gamma\text{-formula } \varphi \). Then for each \( \Gamma\text{-formula } \varphi \) there is a formula in prenex form \(^8\) \( \varphi’ \) such that \( T \vdash \forall \varphi \equiv \varphi’ \).

Lemma 2.36 (Hájek [78]). Let \( * \) be a continuous t-norm. Then \( L_{\forall} \) proves both \((C \exists)\) and \((C \forall)\) iff the \( * \) is the Łukasiewicz’ s t-norm.

In the end of this subsection we review an interesting t-fuzzy logic, resulting from Łukasiewicz logic by adding27 several important formulas which are (non-)provable (see Table 5) in particular predicate t-fuzzy logics. We give their list with their usual names:

Definition 2.37. The Rational Pavelka logic (denoted RPL) is the expansion of Łukasiewicz logic by truth constants \( \bar{r} \) for each rational \( r \in [0, 1] \) and by the bookkeeping axioms for each rational \( r, s \in [0, 1] \):

\[
\bar{r} \& \bar{s} \equiv \bar{r} \& \bar{s}, \quad \bar{r} \rightarrow \bar{s} \equiv \bar{r} \Rightarrow \bar{s}.
\]

The standard RPL-algebra is the standard MV-algebra with \( \bar{r} \) interpreted as \( r \), the finite strong standard completeness of RPL was proved in [66]. Clearly RPL is a core fuzzy logic and so its predicate version RPL\( \forall \) is strongly complete w.r.t. all safe interpretations over all RPL-chains (due to Theorem 2.25).

---

\(^{8}\) i.e., a formula starting with a string of quantifiers followed by a quantifier-free formula.

\(^{9}\) Analogous expansions of other t-fuzzy logics are recently thoroughly studied by the Catalan school of mathematical fuzzy logic, see Section 2.6 for more details. At the end of introduction we mentioned a different approach to this logic, the so-called logic with evaluated syntax (see [124]).
Theorem 2.38 (Hájek et al. [90]). The logic RPLŁ extends Łukasiewicz predicate fuzzy logic ŁŁ conservatively.

Now we turn to the standard semantics: consider only models over the standard algebra RPL-algebra (s-models).

Definition 2.39. Let T be a theory and φ a formula. The provability degree of φ in T is the number

\[ |\varphi|_T = \sup \{ r | T \vdash \varphi \} \]

5 The truth degree of φ in T is the number

\[ \| \varphi \|_T = \inf \{ \| \varphi \|_M \mid M \text{ an s-model of } T \text{ and } v \text{ an } M\text{-evaluation} \} \]

7 Thus the provability degree is the supremum of all r such that T proves that φ is at least r-true; the truth degree is

Theorem 2.40 (Pavelka-style completeness). Let T be a theory over RPLŁ and φ a formula. Then

\[ |\varphi|_T = \| \varphi \|_r \]

This was proved by Novák [118], generalizing Pavelka’s corresponding proof for propositional logic (but for logics

with truth constants for all real numbers \( r \in [0, 1] \), this exact version was proved in [66]). This result can be extended

to the logic PL (see [97]). Continuity of the truth function of implication is heavily used, therefore analogous results

cannot be obtained for almost all other known fuzzy logics; exceptions are of the expansions of Łukasiewicz logic by

continuous connectives like divisions by n (the logic DMV [55]) or algebraic product (the logic PL [97]).

Notice that the Pavelka-style completeness of the logics RPLŁ does not entail its standard completeness. We only

know that a standard tautology φ has provability degree 1, i.e., for each \( r < 1 \) we can prove \( \bar{r} \rightarrow \varphi \).

2.5. A game-theoretic semantics for Łukasiewicz logic

In this subsection we briefly survey a game-theoretic semantics for predicate Łukasiewicz logic. In particular, we deal with special model-theoretic games called evaluation games. There are other game-theoretic semantics for fuzzy logic, introduced by Fermüller in [48]; that approach is proof-theoretic (dialogue games). We also mention Mundici’s semantics for Łukasiewicz propositional logic based on Ulam games with lies [115] and his recent generalization (joint work with Cicallese [28]) for (some) other t-fuzzy (propositional) logics. The reader can find basic information on game theory and evaluation games for the classical logic in particular in [93,130]. This subsection is based on paper [36] where the generalization of these games for Łukasiewicz logic is described.

Let \( B = (B, \ominus, \neg, 0) \) be an MV-chain \(^{10}\) and \( M \) a B-structure. Then the (B, M)-game \( (\varphi, v, r) \) is a zero-sum game of two players Eloise and Abelard which during the course of the game assume roles of Opponent \( O \) and Proponent \( P \). Its states are given by a formula \( \varphi \), an \( M\)-evaluation \( v \), an element \( r \in B \) and assignment of roles to players. \(^{11}\)

Notice that we do not need to assume that \( M \) is a safe \( L\)-structure. The rules of the game are:

\( (at) \ (\varphi, v, r) \), where \( \varphi \) is an atomic formula: \( P \) wins iff \( \| \varphi \|_M \geq r \).

\( (0) \ (\varphi, v, 0); P \) wins.

\( (\oplus) \ (\psi_1 \oplus \psi_2, v, r); P \) chooses \( r' \leq r \) and \( O \) chooses whether to play \( (\psi_1, v, r') \) or \( (\psi_2, v, r \ominus r') \).

\( (\forall) \ (\psi_1 \forall \psi_2, v, r); P \) chooses whether to play \( (\psi_1, v, r) \) or \( (\psi_2, v, r) \).

\( (\exists) \ (\psi_1 \exists \psi_2, v, r); P \) chooses \( r' \leq -r \) and \( O \) chooses whether to play \( (\psi_1, v, r \ominus r') \) or \( (\psi_2, v, r \ominus (r \ominus r')) \).

\( (\wedge) \ (\psi_1 \wedge \psi_2, v, r); O \) chooses whether to play \( (\psi_1, v, r) \) or \( (\psi_2, v, r) \).

\( (\neg) \ (\neg \psi, v, r); O \) chooses \( r', r \geq r' > 0 \), the players switch their roles, game continues as \( (\psi, v, \neg r \ominus r') \).

\( (\forall) \ ((\forall x)\psi, v, r); O \) chooses \( a \in M \), game continues as \( (\psi, v[x \rightarrow a], r) \).

\( (\exists) \ ((\exists x)\psi, v, r); O \) chooses \( r' \neq 0 \) and \( P \) chooses \( a \in M \), the game continues as \( (\psi, v[x \rightarrow a], r \ominus r') \).

\(^{10}\) The MV-algebras are term-wise equivalent with the algebras corresponding to the Łukasiewicz logic, for details see [29].

\(^{11}\) We assume that Eloise is the Proponent in the initial state of the game.

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Again we stress that the rules work also for non-safe structures. The fuzzy evaluation games are zero sum games of a finite depth, so by the Zermelo theorem they are determined. We can prove the correspondence between the existence of winning strategies in a fuzzy game and standard validity.

**Theorem 2.41.** Let $\mathbf{B}$ be an MV-chain, $\mathbf{M}$ a safe $\mathbf{B}$-structure, $\varphi$ a formula, $r \in \mathbb{L}$, and $v$ an $\mathbf{M}$-valuation. Then Eloise has a winning strategy for the $(\mathbf{B}, \mathbf{M})$-game $(\varphi, v, r)$ if and only if $\|\varphi\|^\mathbf{B}_\mathbf{M}, v \geq r$.

It immediately follows that $(\mathbf{B}, \mathbf{M}) \models \varphi[v]$ if and only if Eloise has a winning strategy for the $(\mathbf{B}, \mathbf{M})$-game $(\varphi, v, 1)$. We define two sets induced by the (non)-existence of winning strategies. We use these sets for analyzing the safeness of $\mathbf{B}$-structures.

**Definition 2.42.** Let $\mathbf{B}$ be an MV-chain, $\mathbf{M}$ a $\mathbf{B}$-structure, $\varphi$ a formula, and $v$ an $\mathbf{M}$-valuation. We define:

- $\mathcal{WS}^+(\mathbf{M}, \mathbf{B}, v, \varphi) = \{ r \mid \text{Eloise has a winning strategy for the } (\mathbf{B}, \mathbf{M})\text{-game } (\varphi, v, r) \}$
- $\mathcal{WS}^-(\mathbf{M}, \mathbf{B}, v, \varphi) = \{ r \mid \text{Abelard has for any } r' > r \text{ a winning strategy for the } (\mathbf{B}, \mathbf{M})\text{-game } (\varphi, v, r') \}$

The definition of $\mathcal{WS}^-$ seems to be slightly more complicated than necessary; we need it to obtain a duality of the operations over $\mathcal{WS}^+$ and $\mathcal{WS}^-$ (see [36]). Let us note that there is no game $(\varphi, v, r', r)$ for $r' > r = 1$, so it is always the case that $1 \in \mathcal{WS}^-(\mathbf{M}, \mathbf{B}, v, \varphi)$.

Roughly speaking $r \in \mathcal{WS}^+(\mathbf{M}, \mathbf{B}, v, \varphi)$ whenever the ‘value’ $\varphi$ (in the $\mathbf{B}$-structure $\mathbf{M}$ and $\mathbf{M}$-evaluation $v$) is more than or equal to $r$ and $r \in \mathcal{WS}^-(\mathbf{M}, \mathbf{B}, v, \varphi)$ whenever ‘value’ $\varphi$ is not strictly more than $r$ (we have seen that in safe structures this idea fits perfectly). Thus if $\mathcal{WS}^+(\mathbf{M}, \mathbf{B}, v, \varphi) \cap \mathcal{WS}^-(\mathbf{M}, \mathbf{B}, v, \varphi) = \{ a \}$ we could call $a$ the ‘game-theoretic truth value’ of $\varphi$. Again, we know that in safe structures this new notion coincides with the usual one (due the Theorem 2.41). However, it ‘works well’ in non-safe structures as well, as demonstrated by the following example.

**Example 2.43.** Let $\mathbf{B}$ be the MV-algebra with the domain $[0, 1] \cap \mathbb{Q}$ and the operations defined as usual. Let $q$ be an irrational number and let $a_i$ be a sequence of rationals descending to $q$. Let $\mathbf{M}$ be the $\mathbf{B}$-structure of a predicate language with one unary predicate $P$, where the domain of $\mathbf{M}$ is the set of natural numbers and $P_M(i) = a_i$. Obviously $\mathbf{M}$ is not a safe $\mathbf{B}$-structure— the truth value of $\forall x P(x)$ is undefined. Also the truth value of $\varphi = (\forall x)P(x) \& (\forall x)P(x) \rightarrow (\forall x)P(x)$ is undefined. We can show that $\mathcal{WS}^+(\mathbf{M}, \mathbf{B}, v, \varphi) = [0, 1]$ and $\mathcal{WS}^-(\mathbf{M}, \mathbf{B}, v, \varphi) = \{ 1 \}$ and so $\mathcal{WS}^+(\mathbf{M}, \mathbf{B}, v, \varphi) \cap \mathcal{WS}^-(\mathbf{M}, \mathbf{B}, v, \varphi) = \{ 1 \}$. Thus the ‘game-theoretic truth value’ of $\varphi$ is $1$, as one would expect, although the ‘real’ truth value of $\varphi$ is undefined.

Of course this is just a particular example (e.g., the game-theoretical truth value of the formula $(\forall x)P(x) \rightarrow (\forall x)P(x)$ would we undefined as well). A detailed elaboration of this alternative notion of truth value is a subject of future research.

We conclude this subsection by presenting a theorem giving a game-theoretic characterization of the notion of a safe structure. Notice that the right side of the equivalence does not speak about infima and suprema at all.

**Theorem 2.44.** Let $\mathbf{B}$ be an MV-chain and $\mathbf{M}$ be a $\mathbf{B}$-structure. Then the $\mathbf{B}$-model $\mathbf{M}$ is safe if and only if $\mathcal{WS}^+(\mathbf{M}, \mathbf{B}, v, \varphi) \cap \mathcal{WS}^-(\mathbf{M}, \mathbf{B}, v, \varphi) \neq \emptyset$ for each $v$ and $\varphi$.
3. Variants of $(\triangle)$-core predicate fuzzy logics

In this section we generalize the notion of predicate fuzzy logics we introduced in the previous section in four ways. In more details:

- First, we remove axiom $\forall y$ and show that the resulting logic $L^b \forall y$ is complete w.r.t. all $L$-algebras.  
- Second, we extend the set of logical symbols by the equality symbol, generalize the notion of model accordingly and define the logic $L^= \forall$. 
- Third, we add two more axioms and show the completeness of the resulting logics $L^w \forall$ w.r.t. the so-called witnessed models. 
- Finally, we study fragments of $(\triangle)$-core fuzzy logics where the connective $\lor$ is not definable (i.e., we cannot formulate the axiom $\forall y$). Although such logics fall outside the scope of $(\triangle)$-core fuzzy logics (they do not expand MTL) we can provide axiomatizations of their predicate versions complete w.r.t. their corresponding linearly ordered algebras.

Of course we can combine some of our four proposed generalizations and obtain logics (or their fragments) like $L^w \forall= \forall b$. The syntactical notions for these generalizations (Local Deduction Theorem, Delta Deduction Theorem, Proof by Cases Property, Semilinearity Property, (dubly, $\exists$-)Henkin theory, and linear and directed theory) are defined in the same way as in the previous section. The semantical notions (model, exhaustive model, etc.) are defined also analogously (though we have to alter some of them slightly in the particular cases—see below).

3.1. Logics without $\forall y$

This generalization was proposed in [86]. We start by the definition:

Definition 3.1. Let $L$ be a $(\triangle)$-core fuzzy logic. The logic $L^b \forall y$ has the axioms:

(P) the axioms resulting from the axioms of $L$ by the substitution of propositional variables by $\Gamma$-formulas,

(V1) $(\forall x)\phi(x) \rightarrow \phi(t)$, where $t$ is substitutable for $x$ in $\phi$,

(V1) $(\forall x)\phi(x) \rightarrow (x \rightarrow (\forall x)\phi)$, where $x$ is not free in $\phi$,

(V2) $(\forall x)(\forall \phi \rightarrow \phi) \rightarrow (x \rightarrow (\forall x)\phi)$, where $x$ is not free in $\phi$.

The deduction rules are those of $L$ and generalization: from $\phi$ infer $(\forall x)\phi$.

Of course, the logics $L \forall y$ and $L^b \forall y$ coincide iff $L \forall y$ has $(\forall y)$-eliminability (see Definition 2.29), for examples of particular $t$-fuzzy logics with this property see Tables 3 and 4. All theorems listed in Theorem 2.26 hold in the logic $L^b \forall y$ for $L$ being an arbitrary $(\triangle)$-core fuzzy logic.

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12 The last two points rise an interesting question under which condition does the extension of $L \forall y$ by axiom $(T)$ enjoy such completeness.

13 Therefore we use the superscript $b$ as the logic $L^b \forall y$ is the base (the weakest) predicate logic over a propositional logic $L$. In the existing literature the superscript $b$ is sometimes used to suggest the origin of the logic by removing an axiom from $L \forall y$.

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Now we formulate theorems analogous to Theorems 2.20 and 2.21. Notice that we are not able to prove linearity of the obtained supertheory $T'$.

**Theorem 3.2.** Let $L$ be a core predicate fuzzy logic, $\Gamma$ a predicate language, $T$, $\Psi$ theories such that $T \nvdash \Psi$ and $\Psi$ is directed. Then there is a doubly Henkin $\Gamma'$-theory $T'$ such that $\Gamma \subseteq \Gamma'$, $T \subseteq T'$, and $T' \nvdash \Psi$.

**Theorem 3.3.** Let $L$ be a $\Delta$-core predicate fuzzy logic, $\Gamma$ a predicate language, $T$, $\Psi$ theories such that $T \nvdash \Psi$ and $\Psi$ is directed. Then there is a Henkin theory $T'$ such that $\Gamma \subseteq \Gamma'$, $T \subseteq T'$, and $T' \nvdash \Psi$.

Recall that now we can use Lemma 2.24 to obtain the canonical model only, notice again that we cannot use the first part of this lemma to obtain the linearity of this model. Thus we cannot prove completeness w.r.t. $\ell$-models in general (for a counterexample see [86]). We only obtain:

**Theorem 3.4 (Completeness Theorem).** Let $L$ be a $(\Delta)$-core fuzzy logic, $\Gamma$ a predicate language, $T$ a theory, and $\phi$ a formula. Then the following are equivalent:

1. $T \vdash \phi$.
2. $(B, M) \models \phi$ for each model $(B, M)$ of the theory $T$.
3. $(B, M) \models \phi$ for each exhaustive model $(B, M)$ of the theory $T$.

Note that the present completeness theorem is in fact a particular case of the completeness theorems for the predicate calculi of the so-called implicative logics studied in the early book [127] by Rasiowa.

### 3.2. Logics with crisp equality

In this subsection we add crisp equality to the language and alter the corresponding definitions.

**Definition 3.5.** Let $L$ be a $(\Delta)$-core fuzzy logic and $B$ an $L$-algebra.

1. We extend the set of logical symbols (see Definition 2.1) by a binary predicate $\equiv$.
2. We extend the definition of a formula (Definition 2.3) by a clause that if $r$ and $s$ are terms, then $r = s$ is an atomic formula.
3. We extend the definition of truth value (Definition 2.13) by the condition $\|r = s\|_{M, v}^B = 1$ if $\|r\|_{M, v}^B = \|s\|_{M, v}^B$ and $\|r = s\|_{M, v}^B = 0$ otherwise.
4. The logic $L \forall_w$ (see Definition 2.7) results from $L \forall$ by adding the axioms:

$$=(1) \ (x = y) \lor \neg(x = y)$$

$$=(2) \ x = x$$

$$=(3) \ x = y \rightarrow (\phi(x, \bar{z}) \equiv \phi(y, \bar{z})), \text{ where } y \text{ is substitutable for } x \text{ in } \phi.$$ Observe that in $\Delta$-core fuzzy logics we can remove the first axiom if we replace the third one with:

$$=(3_\Delta) \ x = y \rightarrow \Delta(\phi(x, \bar{z}) \equiv \phi(y, \bar{z})), \text{ where } y \text{ is substitutable for } x.$$

As in classical logic, we can replace the last axiom with the following ones (for each $n$-ary predicate $P$ and $n$-ary function $f$):

$$=(3a) \ x = y \rightarrow y = x$$

$$=(3b) \ (x = y) \& (y = z) \rightarrow x = z$$

$$=(3c) \ (x_1 = y_1) \& \ldots \& (x_n = y_n) \rightarrow (P(x_1, \ldots, x_n) \equiv P(y_1, \ldots, y_n))$$

$$=(3d) \ (x_1 = y_1) \& \ldots \& (x_n = y_n) \rightarrow (f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n))$$

Obviously, we easily get the completeness theorem w.r.t. linear models.

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Theorem 3.6. Let \( L \) be a \((\Delta-)\) core fuzzy logic, \( T \) a theory, and \( \varphi \) a formula. Then the following are equivalent:

1. \( T \vdash_{L_{\varphi}} \varphi \).
2. \( (B, M) \vDash \varphi \) for each (exhaustive) \( \ell \)-model \( (B, M) \) of the theory \( T \).

Analogously we can add \( = \) to the predicate logic \( L_{\varphi} = \) and get the completeness theorem w.r.t. all models.

3.3. Witnessed logics

This subsection is based on paper [84]. Recall that the truth degree of a universally quantified formula is defined as the infimum of truth degrees of its instances and similarly for existentially quantified formula (supremum). The infimum may be smaller than the truth value of each instance (they do not have a minimum); dually for supremum (maximum).

Definition 3.7. We call a formula \((\exists x) \varphi\) containing free variables \( y_1, \ldots, y_n \) witnessed in \((B, M)\) if for each evaluation \( a_1, \ldots, a_n \in M \) of \( y_1, \ldots, y_n \) there is an element \( b \in M \) such that

\[
\| (\exists x) \varphi(x, a_1, \ldots, a_n) \|^{(B, M)} = \| \varphi(b, a_1, \ldots, a_n) \|^{(B, M)};
\]

similarly for \((\forall x) \varphi\). We call model \((B, M)\) witnessed if each formula beginning with a quantifier is witnessed in \((B, M)\).

The notion of a witnessed model was introduced in [74], see also [78,79,83]. Recall axioms \((C\exists), (C\forall), (C\Delta \exists),\) and \((C\Delta \forall)\) introduced in Section 2.4.

Definition 3.8. Let \( L \) be a core fuzzy logic. We define the logic \( L_{\varphi}^w \) as the extension of \( L_{\varphi} \) by axiom schemas \((C\exists), (C\forall)\). Let \( L \) be a \( \Delta \)-core fuzzy logic. We define the logic \( L_{\varphi}^w \) as the extension of \( L_{\varphi} \) by axiom schemas \((C\Delta \exists), (C\Delta \forall)\).

As shown in Table 5 in Section 2.4 the logics \( L_{\varphi} \) and \( L_{\varphi}^w \) sometimes coincide (e.g., in for \( \mbox{\L}ukasiewicz logic). Evidently, \((B, M)\) is witnessed iff all instances of \((C\Delta \exists), (C\Delta \forall)\) are true in \((B, M)\). Thus from the completeness theorem of \( L_{\varphi} \) we immediately get the following:

Theorem 3.9 (Witnessed completeness). Let \( L \) be a \((\Delta-)\) core fuzzy logic, \( \Gamma \) be a predicate language, \( T \) a theory, and \( \varphi \) a formula. Then \( T \vdash_{L_{\varphi}^w} \varphi \) iff \( (B, M) \vDash \varphi \) for each witnessed \( \ell \)-model \( (B, M) \) of the theory \( T \).

The question whether the analog of the previous theorem holds for core fuzzy logics is not trivial. Obviously, if \((B, M)\) is witnessed, then all instances of \((C\exists), (C\forall)\) are true in \((B, M)\); but the converse claim is not valid in general. To show that take any non-witnessed model over standard MV-algebra, e.g., \((\mathbb{N}, r_P)\) where \( r_P(n) = 1/(n + 1) \) (for \( \forall \)) of \( r_P(n) = n/(n + 1) \) (for \( \exists \)). However, we can prove:

Theorem 3.10. Let \( \Gamma \) be a predicate language, \( L \) be a core fuzzy logic, and \((B, M)\) an exhaustive model of a theory \( T \). Then \((B, M)\) is a model of \( T \) over the logic \( L_{\varphi}^w \) iff it can be elementarily embedded into a witnessed \( \ell \)-model of \( T \).

For the definition of elementary embedding see Definition 5.3. Now we easily get the witnessed completeness theorem for core fuzzy logics.

Theorem 3.11 (Witnessed completeness). Let \( L \) be a core fuzzy logic, \( \Gamma \) a predicate language, \( T \) a theory, and \( \varphi \) a formula. Then \( T \vdash_{L_{\varphi}^w} \varphi \) iff \((B, M) \vDash \varphi \) for each witnessed \( \ell \)-model \( (B, M) \) of the theory \( T \).

All what we have done in this subsection could be rephrased for the predicate logics with crisp equality, i.e., we can define logic \( L_{\varphi}^{w_2} \) and prove analogs of Theorems 3.9 and 3.11.
3.4. Fragments of core fuzzy logics

In the paper [34] the authors study fragments of some core fuzzy logics. It turns out that the max-disjunction $\lor$ is not definable in many these fragments by a single formula. Thus the main question for us here is: How do we rephrase the last predicate axiom ($\forall 3$) to the language without (definable) max-disjunction $\lor$?

We restrict ourselves to core fuzzy logics and propositional languages contained in the language of MTL which contains implication (generalization to other languages containing implication would be straightforward). The following theorem tells us that fragments of many core fuzzy logics are in fact axiomatic extensions of corresponding fragment of MTL.

**Theorem 3.12.** Let $\mathcal{L}$ be an axiomatic extension of MTL and $\mathcal{L}$ a propositional language. Then $\mathcal{L}\upharpoonright\mathcal{L}$ is an axiomatic extension of MTL $\downarrow\mathcal{L}$.

In [34] the authors show explicit axiomatization of any fragment of MTL and other fuzzy logics like Łukasiewicz, Product, Gödel, and BL.

**Definition 3.13.** Let $\mathcal{L}$ be a propositional language and $\mathcal{L}$ an axiomatic extension of MTL $\downarrow\mathcal{L}$. The logic $\mathcal{L}\forall'$ has the same axioms as $\mathcal{L}\forall$ with axiom ($\forall 3$) replaced by

$$\forall (\forall x)((\lor \rightarrow \phi) \rightarrow [\forall x((\phi \rightarrow v) \rightarrow v) \rightarrow ((\forall x)\phi \rightarrow v) \rightarrow v])$$

where $x$ is not free in $\chi$.

**Theorem 3.14.** Let $\mathcal{L}$ be a propositional language and $\mathcal{L}$ an axiomatic extension of MTL $\downarrow\mathcal{L}$. Then $\mathcal{L}\forall'$ has Local Deduction Theorem and Semilinear Property.

The following theorem is proven in [34] and it is formulated for at most countable predicate languages only.

**Theorem 3.15 (Completeness theorem).** Let $\mathcal{L}$ be a propositional language, $\mathcal{L}$ an axiomatic extension of MTL $\downarrow\mathcal{L}$, $T$ a theory, and $\phi$ a formula. Then $T\vdash_{\mathcal{L}\forall'} \phi$ iff for each $\mathcal{L}$-chain $\mathcal{B}$ and each $\mathcal{B}$-model $\mathcal{M}$ of $T$ we have $(\mathcal{B}, \mathcal{M})\models\phi$.

Now we can use this completeness theorem together with the completeness theorem for logics with $\lor$ in the language and get the promised result:

**Theorem 3.16.** Let $\mathcal{L}$ be an axiomatic extension of MTL. Then the logics $\mathcal{L}\forall$ and $\mathcal{L}\forall'$ coincide.

It is shown in [34] that (nearly all) fragments of propositional fuzzy logics mentioned in this paper enjoy some form of standard completeness. The question whether some fragments of predicate fuzzy logics enjoy some form of standard completeness as their propositional counterparts seems to be open.

4. Arithmetical complexity of standard and general semantics

In the present section we survey results of the complexity of the several important sets of formulas (like general/standard tautologies, satisfiable formulas, etc., see the formal definition below) for various t-fuzzy logics $\mathcal{L}\forall$ in the sense of the arithmetical hierarchy of sets of natural numbers (or sets of things coded by natural numbers, like formulas, proofs, etc.). We shall restrict ourselves mainly to logics of continuous t-norms (extensions of $BL\forall$). Briefly and roughly said, the reader will see that the sets referring to general semantics are as complex as the corresponding sets given by classical predicate calculus, whereas the sets referring to standard semantics are of varying complexity, some of them being extremely complex (extremely undecidable).

The reader is assumed to know the basic notions of arithmetical hierarchy (see e.g., [129]) and thus we only very briefly remind that a set $X \subseteq N$ is $\Sigma^0_1$ if there is a recursive relation $R \subseteq N^2$ such that $X = \{m|\exists k(R(m,k))\}$. $X$ is $\Pi^0_2$ if there is a recursive relation $R \subseteq N^3$ and $X = \{m|\forall k_1(\exists k_2(R(m,k_1,k_2)))\}$. Similarly for $\Sigma^0_n, \Pi^0_n, n \geq 1$ and $\Pi^0_n, n \geq 1$ (block of alternating quantifiers beginning with $\exists$ or $\forall$, respectively). Let $A$ be $\Sigma^0_n$ or $\Pi^0_n$; $X$ is $A$-hard if each $A$-set
Y is recursively reducible to X, i.e., for a suitable recursive function \( f \), \( Y = \{ k \mid f(k) \in X \} \). X is \( \Lambda \)-complete if \( X \in \Lambda \) and X is \( \Lambda \)-hard. A set X is arithmetical if for some \( n \) \( X \in \Sigma_n \). The set of tautologies of the classical predicate logic is \( \Sigma_1 \)-complete and the set of classically satisfiable formulas is \( \Pi_1 \)-complete. The set \( T h(\mathbb{N}) \) of all formulas of Peano arithmetic true in the standard classical model \( \mathbb{N} \) of arithmetic is a well-known example of a non-arithmetical set.

**Definition 4.1.** Let \( \varphi \) be a sentence of \( L \).

1. \( \varphi \) is a general/standard tautology of \( L \) if \( \| \varphi \|^M = 1 \) for each \( \ell/s \)-model \((B, M)\).
2. \( \varphi \) is generally/standardly satisfiable in \( L \) if \( \| \varphi \|^M = 1 \) for some \( \ell/s \)-model \((B, M)\).
3. We define positive variants of the above four notions (general/standard positive tautologies and generally/standardly positively satisfiable formulas and) by replacing \( \| \varphi \|^M = 1 \) by \( \| \varphi \|^M > 0 \);
   - The classes of above defined formulas are denoted \( t\text{TAUT}(L) \), \( t\text{TAUT}_{pos}(L) \), \( t\text{SAT}(L) \), and \( t\text{SAT}_{pos}(L) \) for \( t \in \{\text{st}, \text{gen}\} \).

**Lemma 4.2.** Let \( L \) be a consistent \((\Delta-)\) core fuzzy logic and \( t \in \{\text{st}, \text{gen}\} \). Then:

\[
\varphi \in t\text{SAT}_{pos}(L) \iff \varphi \notin t\text{TAUT}(L) \\
\varphi \in t\text{TAUT}_{pos}(L) \iff \varphi \notin t\text{SAT}(L)
\]

If the logic \( L \) has an involutive negation (cf. Theorem 2.31) \( \sim \), then also:

\[
\varphi \in t\text{SAT}(L) \iff \sim \varphi \notin t\text{TAUT}(L) \\
\varphi \in t\text{TAUT}(L) \iff \sim \varphi \notin t\text{SAT}(L)
\]

If \( L \) is a \((\Delta-)\) core fuzzy logic, then also:

\[
\varphi \in t\text{SAT}(L) \iff \Delta \varphi \notin t\text{TAUT}(L) \\
\varphi \in t\text{TAUT}(L) \iff \Delta \varphi \notin t\text{SAT}(L)
\]

Recall that for of a class \( t \)-algebras \( \mathcal{K} \) we denote the logic axiomatized by all \( \mathcal{K} \)-tautologies and Modus Ponens as \( L_{\mathcal{K}} \) (we use \( L_{\mathcal{K}} \) instead of \( L_{\{0,1\}_c} \)). Also recall that there is a continuous \( t \)-norm \( * \) such that \( L_{\mathcal{K}} = BL \) (see [1]).

**4.1. Tautologies and satisfiable formulas**

We shall give a rather short survey; for more information the reader is referred to the survey paper [72] and references thereof. First we discuss the general semantics.

**Theorem 4.3.** Let \( * \) be a continuous \( t \)-norm. Then \( \text{genTAUT}(L_{\mathcal{K}}) \) is a \( \Sigma_1 \)-complete set and \( \text{genSAT}(L_{\mathcal{K}}) \) is a \( \Pi_1 \)-complete set.

This theorem was proved for \( \check{\Lambda} \)ukasiewicz, Gödel, product logic and for \( BL \) in [67] and for arbitrary \( * \) in [72].

For standard semantics the situation is different. Before we formulate the results let us mention that for any continuous \( t \)-norm \( * \) different from Gödel \( t \)-norm there are infinitely (uncountably) many continuous \( t \)-norms \( ^* \) isomorphic to \( * \) (hence uncountably many \( t \)-algebras \( [0,1]_* \) isomorphic to \( [0,1]_* \)); for our notions of standard tautologicity/satisfiability it is the same if you consider \( [0,1]_* \) to be the only standard algebra of the corresponding \( t \)-fuzzy logic or consider all \( t \)-algebras isomorphic to \( [0,1]_* \) to be standard algebras.

**Theorem 4.4.**

1. The set \( sT\text{TAUT}(GV) \) is \( \Sigma_1 \)-complete (and equals to \( \text{genTAUT}(GV) \)); the set \( sT\text{TAUT}(L) \) is \( \Pi_2 \)-complete.
2. For each set \( \mathcal{K} \) of continuous \( t \)-norms containing a \( t \)-norm different from Gödel \( t \)-norm, \( sT\text{TAUT}(L_{\mathcal{K}}) \) is \( \Pi_2 \)-hard.
3. Moreover, if \( \mathcal{K} \) is a non-empty set of continuous \( t \)-norms containing a \( t \)-norm which has a \( \check{\Lambda} \)ukasiewicz component which is neither first nor last or a product component, then \( sT\text{TAUT}(L_{\mathcal{K}}) \) is not arithmetical.
For (1) see [66] (the result on Łukasiewicz logic is due to Ragaz); for (2) and (3) see [110]. Note that the arithmetical complexity of \( \text{stTAUT}(L_{\mathcal{K} \forall}) \) for \( \ast \) being one of the t-norms
\[
L \oplus L, G \oplus L, L \oplus G, L \oplus G \oplus L
\]
remains to be an open problem. Now we turn to standard satisfiability.

**Theorem 4.5.**

1. \( \text{stSAT}(G \forall) \) and \( \text{stSAT}(L \forall) \) is \( \Pi_1 \)-complete.
2. If \( \ast \) is a continuous t-norm whose first component is Gödel then we have: \( \text{stSAT}(L_{\mathcal{K} \forall}) = \text{stSAT}(G \forall) \) and hence is \( \Pi_1 \)-complete. Similarly for \( \ast \) whose first component is Łukasiewicz: \( \text{stSAT}(L_{\mathcal{K} \forall}) = \text{stSAT}(L \forall) \).
3. If \( \mathcal{K} \) is a non-empty set of continuous t-norms containing the product t-norm or a t-norm whose first component is product, then \( \text{stSAT}(L_{\mathcal{K} \forall}) \) is not arithmetical.

For (1) see [66]; for (2) see [71]; for (3) see [109]. Observe that the complexity of \( \text{stTAUT}(L_{\mathcal{K} \forall}) \) for continuous t-norms \( \ast \) having no first component is an open problem.

Let us discuss \( \Delta \)-core fuzzy logics. The above theorem on general semantics remains true if you expand the logics by \( \Delta \) (easy). For standard semantics we comment on tautologies.

**Theorem 4.6.**

1. \( \text{stTAUT}(G_{\Delta \forall}) = \text{genTAUT}(G_{\Delta \forall}) \) is \( \Sigma_1 \)-complete.
2. \( \text{stTAUT}(L_{\Delta \forall}) \) is non-arithmetical and so is \( \text{stTAUT}(L_{\mathcal{K} \Delta \forall}) \) for \( \ast \) whose first component is Łukasiewicz.

For (2) see [71]. Observe that combined with the Theorem 4.4 on standard tautologicity (without \( \Delta \)) we see that \( \text{stTAUT}(L_{\mathcal{K} \Delta \forall}) \) is \( \Sigma_1 \)-complete for \( \ast \) being Gödel and is non-arithmetical for all other continuous t-norms except possibly \( G \oplus L \) (for which the precise complexity is unknown). Standard satisfiability of logics with \( \Delta \) has to be investigated.

4.2. Results for ‘positive’ and ‘witnessed’ cases

First we deal with positive tautologies and positively satisfiable formulas. For general semantic the complexity result is the same as above [72, Theorems 5–7]:

**Theorem 4.7.** Let \( \ast \) be a continuous t-norm. Then \( \text{genTAUT}_{\text{pos}}(L_{\ast \forall}) \) is a \( \Sigma_1 \)-complete set and \( \text{genSAT}_{\text{pos}}(L_{\ast \forall}) \) is a \( \Pi_1 \)-complete set.

Now for the standard semantics [72]:

**Theorem 4.8.**

1. Let \( \ast \) stand for Gödel, Łukasiewicz or product and let \( \ast \) a t-norm beginning with the t-norm \( C \). Then \( \text{stTAUT}_{\text{pos}}(C \forall) = \text{stTAUT}_{\text{pos}}(L_{\ast \forall}) \) and \( \text{stSAT}_{\text{pos}}(C \forall) = \text{stSAT}_{\text{pos}}(L_{\ast \forall}) \).
2. \( \text{stTAUT}_{\text{pos}}(G \forall) \) is \( \Sigma_1 \)-complete; \( \text{stSAT}_{\text{pos}}(G \forall) \) is \( \Pi_1 \)-complete. Also, \( \text{stTAUT}_{\text{pos}}(L \forall) \) is \( \Sigma_1 \)-complete and \( \text{stSAT}_{\text{pos}}(L \forall) \) is \( \Sigma_2 \)-complete.
3. If \( \mathcal{K} \) is a non-empty set of continuous t-norms containing a t-norm whose first component is product, then \( \text{stTAUT}_{\text{pos}}(L_{\mathcal{K} \forall}) \) and \( \text{stSAT}_{\text{pos}}(L_{\mathcal{K} \forall}) \) is not arithmetical.

Again here the complexity for logics given by a t-norm without first component remains to be an open problem. Second, we consider the complexity of the sets of general standard (positive) tautologies and generally/standardly (positively) satisfiable sentences of the witnessed variants of our logics as studied in [78,79,83]. The results will not be fully presented here; we just illustrate them by the following theorems (the prefix ‘w-’ means restriction to witnessed models):

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Theorem 4.9. For each $\ast$, $w\text{-}\text{genTAUT}(L_\ast\forall)$ is $\Sigma_1$-complete and $w\text{-}\text{genSAT}(L_\ast\forall)$ is $\Pi_1$-complete.

Theorem 4.10. For $\ast$ with Gödel negation holds:

1. The following five sets of formulas are equal: $w\text{-}\text{genSAT}(L_\ast\forall)$, $w\text{-}\text{genSAT}_{\text{pos}}(L_\ast\forall)$, $w\text{-}\text{stSAT}(L_\ast\forall)$, $w\text{-}\text{stSAT}_{\text{pos}}(L_\ast\forall)$, the set of classically satisfiable formulas.

2. The following sets are equal: $w\text{-}\text{genTAUT}_{\text{pos}}(L_\ast\forall)$, $w\text{-}\text{stTAUT}_{\text{pos}}(L_\ast\forall)$, and the set of classical tautologies (but different from $w\text{-}\text{genTAUT}(L_\ast\forall)$ and $w\text{-}\text{stTAUT}(L_\ast\forall)$).

For result on other t-norm based fuzzy logics see the tables in [78, Section 4].

4.3. Further reading

As the reader could see, several problems remain open and the theory of arithmetical complexity of fuzzy logics remains an interesting domain of current research. Other interesting and/or recent results in this area are summarized in the following list.

- In [72] the reader also finds results on arithmetical complexity concerning falsity-free predicate logics (hoop predicate logics). The paper [80] shows that the sets $\text{genTAUT}$, $\text{genSAT}$, $\text{stTAUT}$ and $\text{stSAT}$ of the $(\rightarrow, \neg)$-fragments of the logics $BL_\forall$, $L_\forall$, $G_\forall$, $II_\forall$ have the same complexity as these logics themselves, e.g., the set of all satisfiable sentences of the $(\rightarrow, \neg)$-fragment of $II_\forall$ is not arithmetical etc. Some proofs are easy but some are not, in particular the results of non-arithmeticity due to the lack of strong conjunction. For standard satisfiability one has to deal with standardly satisfiable finite sets of sentences.

- Results on standard tautologies and standardly satisfiable formulas of Łukasiewicz predicate logic are easily modified to get the same results for Rational Pavelka predicate logic alias Novák’s predicate logic with evaluated syntax.

- Montagna and Ono proved [113] logic $MTL_\forall$ has standard completeness, which implies (together with general completeness of $MTL_\forall$) almost immediately that $\text{stTAUT}(MTL_\forall) = \text{genTAUT}(MTL_\forall)$ is $\Sigma_1$-complete and (by analyzing the proof in Montagna-Ono’s paper) that $\text{stSAT}(MTL_\forall) = \text{genSAT}(MTL_\forall)$ is $\Pi_1$-complete.

- Let us also mention the results of Montagna and Sacchetti [114] showing that when defining the general semantics of our logics we cannot restrict ourselves to $L$-chains that are completely ordered lattices (all sets have infimum and supremum), e.g., a formula is a tautology over all completely ordered MV-algebras iff it is a standard tautology of $L_\forall$, the set of all (predicate) tautologies over completely ordered MV-algebras thus is $\Pi_2$-complete.

- Paper [33], fully solves the arithmetical complexity issues of all axiomatic extensions of Łukasiewicz logic.

- Recent paper [111] provides lower bounds for the complexities associated to arbitrary semantics of arbitrary $(\Delta)$-core fuzzy logics and computes upper bounds and exact positions in the arithmetical hierarchy for particular logics and semantics based on all/finite/real/rational chains.

5. Some model theory

Understanding t-norm based predicate fuzzy logic as a branch of mathematical logic implies interest in a kind of model theory analogous to the model theory of classical predicate calculus. This is a very interesting domain of problems. Here we survey two examples of such results—a model theoretical characterization of conservative extensions of theories and a form of the Birkhoff theorem in fuzzy logic. We start this section by some basic model theoretic definitions from [84].

Definition 5.1 (Diagram). Let $\Gamma$ be a predicate language and $(B, M)$ a model of $\Gamma$. Then we define:

- $\Gamma_M$ is the predicate language resulting from $\Gamma$ by adding an object constant $c_a$ for each $a \in M$
- $M^\ast$ is a $\Gamma_M$-model resulting from $M$ by setting $(c_a)_{M^\ast} = a$ for $a \in M$
- $\text{DIAG}(B, M) = \text{Th}((B, M^\ast))$ (the set of all sentences true in $(B, M^\ast)$)

The set $\text{DIAG}(B, M)$ is called the diagram of the model $(B, M)$.

In [32] the notion of full diagram is introduced, it also adds truth constants (nullary predicates) for all elements of $B$.  

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Definition 5.2 (Elementary equivalence). Let \((B_1, M_1)\) and \((B_2, M_2)\) be two models interpreting the same language. We say that \((B_1, M_1)\) is elementarily equivalent to \((B_2, M_2)\) if for each sentence \(\varphi\) we have: \((B_1, M_1) \vDash \varphi \iff (B_2, M_2) \vDash \varphi\).

Definition 5.3 (Elementary embedding). An elementary embedding of a model \((B_1, M_1)\) of a language \(\Gamma_1\) into a model \((B_2, M_2)\) of a language \(\Gamma_2 \supseteq \Gamma_1\) is a pair \((f, g)\) such that:

1. \(f\) is an injection of \(M_1\) into \(M_2\).
2. \(g\) is an embedding of \(B_1\) into \(B_2\).
3. \(g(\parallel \varphi(a_1, \ldots, a_n)\parallel_{B_1(M_1)}) = \parallel \varphi(f(a_1), \ldots, f(a_n))\parallel_{B_2(M_2)}\) holds for each \(\Gamma_1\)-formula \(\varphi(x_1, \ldots, x_n)\) and \(a_1, \ldots, a_n \in M_1\).

\((B_1, M_1)\) is an elementary submodel of \((B_2, M_2)\) if \(f\) and \(g\) are the identity mappings on the respective domains and \((f, g)\) is an elementary embedding of \((B_1, M_1)\) into \((B_2, M_2)\).

We formulate a lemma stating the connection between elementary embedding and elementary equivalence and the transitivity of the notion of elementary embedding. Then we demonstrate that the definition of an elementary embedding can be simplified in the case that \((B_1, M_1)\) is exhaustive.

Lemma 5.4. If a model \((B_1, M_1)\) can be elementarily embedded into a model \((B_2, M_2)\), then the models \((B_1, M_1)\) and the reduct of \((B_2, M_2)\) to the language of \((B_1, M_1)\) are elementarily equivalent.

If a model \((B_1, M_1)\) can be elementarily embedded into a model \((B_2, M_2)\) and the model \((B_2, M_2)\) can be elementarily embedded into a model \((B_3, M_3)\), then the model \((B_1, M_1)\) can be elementarily embedded into the model \((B_3, M_3)\).

Lemma 5.5. Let \((B_1, M_1)\) be an exhaustive model. Then a pair \((f, g)\) is an elementary embedding of model \((B_1, M_1)\) into a model \((B_2, M_2)\) iff:

1. \(f\) is an injection of \(M_1\) into \(M_2\).
2. \(g\) is an injection of \(B_1\) into \(B_2\).
3. \(g(\parallel \varphi(a_1, \ldots, a_n)\parallel_{B_1(M_1)}) = \parallel \varphi(f(a_1), \ldots, f(a_n))\parallel_{B_2(M_2)}\) holds for each \(\Gamma_1\)-formula \(\varphi(x_1, \ldots, x_n)\) and \(a_1, \ldots, a_n \in M_1\).

5.1. Conservative extensions

In this subsection we present the model-theoretic characterization of the conservativeness of an extension. In classical logic, it is an easy exercise of application of compactness and completeness. However in fuzzy logics we need the full power of the Fundamental Lemma 2.19 to achieve the result (see [84] for details).

We start by two important lemmata. Notice that we formulate them for exhaustive models only. Its extension for all models seems to be an interesting open problem. Also notice that in the case of core fuzzy logics it guarantees the existence of doubly Henkin (and not only Henkin) theory \(T\), which is crucial in proving Theorem 3.11.

Lemma 5.6. Let \(L\) be a \((\Delta-)\) core fuzzy logic. If \(T_2\) is a conservative extension of \(T_1\), then for each exhaustive \(\ell\)-model \((B, M)\) of \(T_1\) there exists a linear Henkin theory \(T\) extending \(T_2\) such that \((B, M)\) can be elementarily embedded into \(CM(T)\).

Lemma 5.7. Let \(L\) be a core predicate fuzzy logic. If \(T_2\) is a conservative extension of \(T_1\), then for each exhaustive \(\ell\)-model \((B, M)\) of \(T_1\) there exists a linear doubly Henkin theory \(T\) extending \(T_2\) such that \((B, M)\) can be elementarily embedded into \(CM(T)\).

Theorem 5.8. Let \(L\) be a \((\Delta-)\) core fuzzy logic and \(T_1, T_2\) theories over \(L\). Then the following claims are equivalent:

1. \(T_2\) is a conservative extension of \(T_1\).
2. Each exhaustive \(\ell\)-model of \(T_1\) can be elementarily embedded into some \(\ell\)-model of \(T_2\).
3. Each exhaustive \(\ell\)-model of \(T_1\) is an elementary submodel of some \(\ell\)-model of \(T_2\).

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4. Each exhaustive \( \ell \)-model of \( T_1 \) is elementarily equivalent to the restriction of some \( \ell \)-model of \( T_2 \) to the language of \( T_1 \).

5. Each \( \ell \)-model of \( T_1 \) is elementarily equivalent to the restriction of some \( \ell \)-model of \( T_2 \) to the language of \( T_1 \).

One can notice that conditions 4 and 5 differ only in the scope of quantification, the former speaks about all exhaustive models and the later about all models. There is an obvious question whether the analogy holds for condition 2 as well, i.e., whether we could equivalently add condition 6:

6. Each \( \ell \)-model of \( T_1 \) can be elementarily embedded into some \( \ell \)-model of \( T_2 \).

This question seems to be rather non-trivial. However, we can prove its equivalence with a much simpler question:

**Theorem 5.9.** Let \( L \) be a \((\Delta\text{-})\)-core fuzzy logic. Then the following claims are equivalent.

- Let \( \Gamma_1 \) and \( \Gamma_2 \) be predicate languages and \( T_i \) be a \( \Gamma_i \)-theory. If \( T_2 \) is a conservative extension of \( T_1 \), then each \( \ell \)-model of \( T_1 \) can be elementarily embedded into some \( \ell \)-model of \( T_2 \).

- Let \( P \) be a nullary predicate symbol, \( \Gamma_1 \) and \( \Gamma_2 \) predicate languages, \( T_i \) a \( \Gamma_i \)-theory, and \( T_i^+ \) a \( \Gamma_i \cup \{ P \} \)-theory such that \( T_i^+ = T_i \) (i.e., \( P \) is added to the language but no new axioms are added). If \( T_2 \) is a conservative extension of \( T_1 \), then \( T_2^+ \) is a conservative extension of \( T_1^+ \).

5.2. The Birkhoff variety theorem

In [22], Bělohlávek develops a kind of fuzzy universal algebra over an arbitrary fixed complete residuated lattice \( L \) and shows that a class of \( L \)-fuzzy algebras (algebras with an \( L \)-fuzzy equality) is a variety (i.e., closed under homomorphic images, subalgebras and direct products) iff it is the class of all models of an \( L \)-fuzzy set of identities. In the book [23] and Vychodil study fuzzy equational logic, in particular they present an axiomatization for the consequence relation \( \Sigma = (\varphi, r) \) (\( \Sigma \) an \( L \)-fuzzy set of identities, \( \varphi \) an identity, \( r \in L \)) and show its Pavelka-style completeness, i.e., degree of provability = degree of semantical consequence (see also their paper on quasiequational consequence [24]).

Needless to say, this is a sound and elegant approach, but not the only one possible. Here we sketch some possibilities of universal algebra over our basic predicate fuzzy logic \( BL\mathcal{V}^b \). We introduce algebras with fuzzy equality, the notions of subalgebra, homomorphic image, direct product and one additional operation \( ske \) of skeleton. All theories (systems of identities) are crisp. We formulate and prove a kind of Birkhoff variety theorem (based on some unpublished results by Hájek).

We deal with a fixed language consisting of one binary predicate \( \approx \) (similarity) and some function symbols \( F_1, \ldots, F_n \), each having some arity. Thus atomic formulas have the form \( \sigma \approx \tau \), where \( \sigma \) and \( \tau \) are terms. Our logic is \( BL\mathcal{V}^b \) with functions and \( \approx \) as equality predicate, i.e., we postulate the equality axioms

\[
(\forall x)(x \approx x) \quad (\text{reflexivity})
\]

\[
(\forall x, y)(x \approx y \to y \approx x) \quad (\text{symmetry})
\]

\[
(\forall x, y, z)((x \approx y \& y \approx z) \to x \approx z) \quad (\text{transitivity})
\]

\[
(\forall x_1, \ldots, x_k, y_1, \ldots, y_k)((x_1 \approx y_1 \& \ldots \& x_k \approx y_k) \to F_i(x_1, \ldots) \approx F_i(y_1, \ldots))
\]

where \( F_i \) is of arity \( k \) (congruence).

For any \( BL \)-algebra \( L \), an \( L \)-fuzzy algebra is a safe \( L \)-interpretation

\[
\mathcal{M} = (M, F_1^M, \ldots, F_n^M, \approx^M)
\]

of this language which is a model of the equality axioms and \( \approx^M \) is a fuzzy equality. (We may assume \( a \approx^M b = 1 \) iff \( a = b \) for all \( a, b \in M \)). The pair \((B, \mathcal{M})\) is just called a fuzzy algebra. \((B, \mathcal{M})\) is crisp if \( L = B_0 \) where \( B_0 \) is the two-element Boolean algebra, then \( \approx^M \) is the characteristic function \( Id_M \) of the identity relation on \( M \). The crisp algebra \((B, B_0)\) where \( B \) is as above may be identified with the classical algebra \((M, F_1^B, \ldots, F_n^B)\).

A fuzzy algebra \((L_1, \mathcal{M}_1)\) is a fuzzy subalgebra of \((L_2, \mathcal{M}_2)\) if \( L_1 \) is a \( BL \)-subalgebra of \( L_2 \) and \( \mathcal{M}_1 \) is a restriction of \( M_2 \) to \( M_1 \), i.e., \( F_i^{M_1}(a_1, \ldots, a_k) = F_i^{M_2}(a_1, \ldots, a_k) \) for \( a_1, \ldots, a_k \in M_1 \) and \( a_1 \approx^{M_1} a_2 = a_1 \approx^{M_2} a_2 \) for \( a_1, a_2 \in M_1 \).
(L₁, M₁) is a homomorphic image\(^{14}\) of (L₂, M₂) if there is a pair of mappings (h_M, h_L) such that h_L is a homomorphism of the BL-algebra L₂ into L₁, h_M is a surjection of M₂ onto M₁ commuting with the F's, i.e.,

\[ h_M(F^L_i(a_1, \ldots, a_k)) = F^M_i(h_M(a_1), \ldots, h_M(a_k)) \]

for all \(a_1, \ldots, a_k \in M_2\), and finally, \(h_L\) respects \(\approx\), i.e.

\[ h_L(a_1 \approx^M_2 a_2) \leq h_M(a_1) \approx^M_1 h_M(a_2) \]

for all \(a_1, a_2 \in M_2\).

(See [21] 3.4.2 and [117].)

Let \(ske(M)\) result from \(M\) by replacing \(\approx^M\) by the identity \(Id_M\) and let the skeleton of \((M, L)\) be defined as \(ske(M, L) = (ske(M), B_0)\) where \(B_0\) is the two-element Boolean algebra. Observe that if \(T\) is a theory whose axioms are just atomic formulas (or their universal closures), then \(M\) is an \(L\)-model of \(T\) iff \(ske(M)\) is a \(B_0\)-model (crisp model) of \(T\).

Finally, if \((M_i, L_i)\) are fuzzy algebras for \(i \in I\), then we define the (fuzzy) direct product \(\Pi_{i \in I}(M_i, L_i)\) as follows:

\[ \Pi_{i \in I}(M_i, L_i) = (\Pi_{i \in I}M_i, \Pi_{i \in I}L_i) \]

where \(\Pi_{i \in I}L_i\) is the direct product of BL-algebras \(L_i\) (in the classical sense; cartesian product of domains, operations coordinatewise) and \(M = \Pi_{i \in I}M_i\) is the \(\Pi_{i \in I}L_i\)-fuzzy algebra whose domain is the cartesian product of the \(M_i\)’s, the \(F_i\) are defined coordinatewise and so is \(\approx^M:\)

\[ \{a_i\}_{i \in I} \approx^M \{b_i\}_{i \in I} = \{a_i \approx^M_i b_i\}_{i \in I}. \]

Observe that if \((M_i, B_0)\) are classical algebras, then their classical direct product is the classical algebra \(ske(\Pi_{i \in I}(M_i, B_0)) = (ske(\Pi_{i \in I}M_i), B_0)\). Clearly, \(ske(M)\) is safe (being crisp).

We keep the language \(F_1, \ldots, F_n, \approx\) fixed. A class \(\forall^c\) of fuzzy algebras is a variety (of fuzzy algebras) iff it is closed under subalgebras, homomorphic images, (fuzzy) direct products and skeletons.

Recall that a classical variety is a class of classical algebras closed under subalgebras, homomorphisms and (classical) direct products. The classical Birkhoff variety theorem says that a class \(\forall^c\) of classical algebras is a classical variety iff there is a set \(T\) of identities (atomic formulas of the form \(\tau \approx \sigma\), where \(\tau, \sigma\) are terms) such that \(\forall^c\) is the class of all (classical) models of \(T\).

Theorem 5.10 (Birkhoff variety theorem for fuzzy algebras). The following are equivalent:

1. \(\forall^c\) is a variety of fuzzy algebras,
2. there is a set \(T\) of identities such that \(\forall^c\) is the class of all \(L\)-models of \(T\), \(L\) being any BL-algebra,
3. (classical) models of \(T\) of \(\forall^c\) is the set of all homomorphic images of elements of \(\forall^c\).

Admittedly, our result concerning the Birkhoff variety theorem over our fuzzy logic is modest: our varieties are just classes of homomorphic images of classical crisp varieties. But recall that we are not forced to work over one fixed BL-algebra; neither are we forced to assume the completeness of the lattice ordering of the algebras. Working with safe structures allows future investigation of classes of algebras which are models of syntactically more complex theories than those given by universally quantified identities (think of e.g., formulas with alternating quantifiers), still without assuming completeness of the order. This keeps us in the domain of general semantics of fuzzy logics. The problem that offers itself immediately is to investigate quasivarieties over fuzzy logic (classically a quasivariety is a class of algebras closed under subalgebras, direct product and direct limits, see e.g., [135]) and its relation to classes of models of a theory \(T\) whose axioms are universal Horn sentences, i.e., formulas \((\forall \ldots)(K \rightarrow \sigma)\) where \(\sigma\) is an identity and \(K\) is a (finite) conjunction of identities. Note that if \((M, L)\) is a model of such \(T\), then \(ske(M, L)\) is also its model but not conversely. Thus some further investigation seems promising.

\[\text{Cf. the notion of elementary embedding from the previous subsection.}\]
5.3. Further reading

Other interesting and/or recent results in the model-theoretic aspects of predicate fuzzy logics are summarized in the following list.

- Let us mention results in the context of the logic with evaluated syntax by Murinová and Novák [116] on omitting types theorem and [119] on joint consistency (see also a chapter on model theory in the book [124]).
- Antonio Di Nola, George Georgescu and Luca Spada in their recent paper [38] develop the notion of forcing for Łukasiewicz predicate logic, along the lines of Robinson’s forcing in classical model theory. They deal with both finite and infinite forcing, prove a Generic Model Theorem for the finite one, and study the generic and existentially complete standard models for the infinite one.
- As mentioned in Section 2, paper [32] characterizes strong completeness w.r.t. arbitrary classes of chains via existence of certain elementary embeddings (see Theorem 2.28). For a given logic L and a class K of L-chains the paper also shows how to strengthen this model theoretic condition to describe when each at most countable non-trivial L-chain can be embedded into some algebra from K in such a way that all existing suprema and infima are preserved by this embedding.

6. Examples of important theories over predicate fuzzy logics

In the preceding sections we have developed predicate t-fuzzy logics and studied the notion of a theory over a predicate t-fuzzy logic. In this section we are going to survey some particular important theories over (some) t-fuzzy logics, namely some kinds of arithmetic and set theory. Besides theories as they were defined in previous sections (as given by a language and a set of special axioms) we shall have to work also with a generalized notion of a theory (over a given logic), i.e., a theory can be given by its language, special axioms and special deduction rules. The notions of a proof and a model are then modified in the obvious way. In the first subsection we shall show how a suitable arithmetic over a t-fuzzy logic may help us understand some paradoxes (the liar paradox and the sorites paradox). Then we present two approaches to set theory in t-fuzzy logic: a fuzzy set theory similar to the classical Zermelo–Fraenkel set theory over a t-fuzzy logic may help us understand some paradoxes (the liar paradox and the sorites paradox). Then we present two approaches to set theory in t-fuzzy logic: a fuzzy set theory similar to the classical Zermelo–Fraenkel set theory and a fuzzy set theory similar to Cantor’s set theory with a full comprehension as possible. 15

6.1. Fuzzy arithmetic

We shall work with Peano arithmetic $PA$ whose language consists of the predicates $=, \leq$, object constant $\bar{0}$ and function symbols $S$ (successor, unary), $+$ (addition, binary) and $\cdot$ (multiplication, binary); the axioms are the finitely many axioms of the Robinson arithmetic

(Q1) $S(x) \neq \bar{0}$.
(Q2) $S(x) = S(y) \rightarrow x = y$.
(Q3) $x \neq \bar{0} \rightarrow (\exists y)(x = S(y))$.
(Q4) $x + \bar{0} = x$.
(Q5) $x + S(y) = S(x + y)$.
(Q6) $x \cdot \bar{0} = \bar{0}$.
(Q7) $x \cdot S(y) = (x \cdot y) + x$.
(Q8) $x \leq y \equiv (\exists z)(z + x = y)$

and the deduction rule of induction: for any formula $\varphi$, from $\varphi(0)$ and from $(\forall x)(\varphi(x) \rightarrow \varphi(S(x)))$ infer $(\forall x)\varphi(x)$. Note that over classical logic the deduction rule of induction is equivalent to the induction schema of axioms

$(\varphi(0)) \&(\forall x)(\varphi(x) \rightarrow \varphi(S(x)))) \rightarrow (\forall x)\varphi(x)$.

(See [91] for the metamathematics of classical first-order arithmetic.) But we shall see that in fuzzy logic we have to postulate the deduction rule of induction and not the induction axiom schema.

15 There are other axiomatic fuzzy set theories, but only the mentioned ones have been developed inside predicate fuzzy logic, see [61,62] for more details.
First take the axioms and the deduction rule of PA as they stand and consider them over the basic logic $BL\forall$. Restall [128] has proved that tertium non datur $\varphi \lor \neg \varphi$ follows for all formulas of the language $PA$, thus we get classical logic. (Restall proves this for Łukasiewicz logic but his proof works in general.) We are interested in adding further predicates, not definable in the language of arithmetic and not necessarily crisp, assuming the rule of induction for the expanded language. Then the axiom schema of induction is no more sound: think of a model expanding the $s$-model of arithmetic by a new unary predicate $P$ having the value $0.5$ for $x = 0$ and the value $0$ for all other arguments; over $\forall \exists$ the induction axiom would give $0.5$ as the lower estimate for the value of $(\forall x)P(x)$, but the value of the last formula is $0$. A similar example is easy to give for $\Pi \forall$; but for $G \forall$ the axiom schema of induction is derivable from the deduction rule.

The liar paradox: the paradoxical sentence stating its own falsity can be constructed in $PA$ expanded with a truth predicate $Tr$ and the axiom schema $\varphi \equiv Tr(\overline{\varphi})$ (dequotation schema), where $\overline{\varphi}$ is the numeral corresponding to the Gödel number of the formula $\varphi$: By the Gödel diagonal lemma there is a formula $\lambda$ such that our theory proves $\lambda \equiv \neg(Tr(\overline{\lambda}))$, which implies $\lambda \equiv \neg \lambda$ and this is a contradiction over classical logic. The construction can be analogously performed over $BL\forall$ (and hence over each stronger logic); over Łukasiewicz logic a formula $\lambda \equiv \neg \lambda$ need not be inconsistent: in every model over $[0,1]_L$ it just must have the truth value $0.5$. (Over all logics with Gödel negation the last formula is contradictory.) The situation for Łukasiewicz logic is as follows:

**Theorem 6.1.** Let $PATr$ be the expansion of $PA$ with the truth predicate $Tr$ and the dequotation schema; let the logic be $\forall \exists$.

(i) The theory $PATr$ is consistent: there is a crisp (classical) model of $PA$ expandable to a $[0,1]_L$-model of $PATr$.

(ii) The standard crisp model of $PA$ (the structure of natural numbers) cannot be expanded to a model of $PATr$—the theory is $\omega$-inconsistent.

(iii) The extension of $PATr$ with the axiom saying that $Tr$ commutes with connectives is inconsistent.

These are the results from [89]; (ii) was proved even earlier by Restall in [128]. For another elegant proof see [137]. Clearly the theorem works with the arithmetization of logic in $PA$; in particular, the axiom formulated in (iii) is a single formula saying for implication

$$(\forall u, v)((Fmla(u) & Fmla(v)) \rightarrow (Tr(u \rightarrow v) \equiv (Tr(u) \rightarrow Tr(v))))$$

and similarly for other connectives. Note that (ii) is proved in [89] using a formula called the ‘modest liar formula’ stating “I am at least a little false”. Finally note that the deduction rule of induction is evidently a consequence of the axiom schema using $\Delta$ and saying

$$\Delta(\varphi(0) & \Delta(\forall x)(\varphi(x) \rightarrow \varphi(S(x)))) \rightarrow \Delta(\forall x)\varphi(x).$$

This might be useful for some expansions of $PA$, but unfortunately adding the truth predicate and dequotation schema to modification of $PA$ over Łukasiewicz logic with $\Delta$ is inconsistent, the inconsistent liar formula being the formula for which $\lambda \equiv \Delta Tr(\neg \lambda)$ is provable. Thus we have to accept the deduction rule of induction.

The paper [77] studies a very weak arithmetic $\forall Q^-$ which is a variant of Robinson’s arithmetic $Q$ where addition and multiplication are described by ternary predicates not assuming that they describe total functions and the underlying logic is fuzzy ($BL\forall$ or similar). The first Gödel-style incompleteness is proved (each axiomatizable consistent extension of $\forall Q^-$ is incomplete) and essential undecidability is claimed, but the proof has a gap. Full proof is in [65].

The sorites paradox: This is also called the heap paradox. One grain of sand does not make a heap and adding one grain to a non-heap does not make it a heap. Thus $2, 3, \ldots, n$ grains do not make a heap, for arbitrary $n$; there are no heaps. For a survey on the sorites paradox see e.g., [98]; here we mention the approach from the paper [88]. The heap paradox (bald man paradox, etc.) deals with a form of the notion of a small natural number. This notion was investigated in an excellent paper by Parikh [125] under the name of a feasible number; we keep this terminology and turn to fuzzy logic. The idea is that for each $n$, the formula $Fe(n)$ (saying that $n$ is feasible) has some truth degree; and that the formula $Fe(n+1)$ has a truth degree which is possibly slightly less than the former truth degree, but not

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16 For detail about formalization of the logical syntax in $PA$ see [91].
much less. This leads us to the idea to extend the language of our fuzzy logic (say, BL∀ or stronger) by a new unary
connective (hedge) At; read the formula At(φ) ‘φ is almost true’. The axioms for At are
\[ \varphi \rightarrow At(\varphi), \; (\varphi \rightarrow \psi) \rightarrow (At(\varphi) \rightarrow At(\psi)). \]

In this logic we shall deal with crisp Peano arithmetic PA (as in the preceding section, i.e., with the deduction rule of
induction) extended with the (fuzzy) predicate Fe and postulate
\[ Fe(0), \; x < y \rightarrow (Fe(y) \rightarrow Fe(x)), \; Fe(x) \rightarrow At(Fe(x + 1)). \]

This is a consistent theory and in [88] the reader can find several examples of possible semantics of the connective
At and examples of expansions of the standard model of arithmetic by an interpretation of the predicate Fe. A simple
example is: let \( p \) be a truth constant for a number less than 1 and let At(φ) mean \( p \rightarrow \varphi \). Let the value \( \| Fe(n) \| \)
be \( p^n \) for your chosen continuous t-norm, i.e., \( p \ast p \ast \ldots \ast p \) (\( n \) factors, \( p^0 \) being 0). You get a \([0,1]^d\)-model \( M \) of our theory.

If your t-norm is Łukasiewicz, then \( (\exists x) \neg Fe(x) \) is true in the model; if you choose product, then \( \neg (\forall x) Fe(x) \) is true.

In [88] there are examples of models satisfying even
\[ (Fe(x) \& Fe(y)) \rightarrow At(Fe(x + y)) \quad \text{and} \quad (Fe(x) \& Fe(y)) \rightarrow At(Fe(x \cdot y)), \]

thus, roughly, ‘if \( x, y \) are feasible, then \( x + y \) and \( x \cdot y \) are almost feasible’, but still the formula \( \neg (\forall x) Fe(x) \) is true.

The paper also discusses the paradox in the frame of fuzzy logic with evaluated syntax.

6.2. A fuzzy set theory extending Zermelo–Fraenkel

The universe of the classical Zermelo–Fraenkel set theory ZF is the union of the hierarchy of sets constructed from
the empty set by the transfinitely iterated operation of power set. Klaua [102,103] and Gottwald [58–60] generalized this
construction to obtain inside ZF a kind of model of a universe of fuzzy sets (without presenting any axiomatic system
of set theory in fuzzy logic). The construction presented below is similar to theirs (and similar to the construction of the
famous Boolean-valued models of set theory). We formulate an axiomatic system FST of fuzzy set theory (within the
logic BL∀) and show that both FST is interpretable in ZF (using the model constructed) and also ZF is interpretable
in FST. FST admits the existence of (proper) fuzzy sets of fuzzy sets of fuzzy sets .... Everything in this subsection is
taken from paper [87], where the reader can find all details.

In ZFC (Zermelo–Fraenkel set theory with the axiom of choice) let \( L \) denote an arbitrary fixed completely ordered
BL-chain and let \( L^+ \) denote the set of elements of \( L \) different from 0. Let \( V^L_0 = \{ \emptyset \} \), for each ordinal \( \alpha \) let \( V^L_{\alpha+1} \) be the
set of all functions mapping any subset of \( V^L_\alpha \) into \( L^+ \) and for any limit ordinal \( \lambda \) let \( V^L_\lambda \) be \( \bigcup \{ V^L_\alpha | \alpha < \lambda \} \). Finally, \( V^L \)
is the union of all \( V^L_\alpha \).

Now define, for any \( x, y \in V^L \),
\[ \| x \in y \| = y(x) \text{ if } x \in dom(y), \text{ and } = 0 \text{ otherwise}; \]
\[ \| x = y \| = 1 \text{ if } x = y, \text{ and } = 0 \text{ otherwise}. \]

This is extended in a natural way to the definition of \( \| \varphi \| \) for each \( \varphi \); if \( \varphi \) has \( n \) free variables, then \( \| \varphi \| \)
is a function from \( (V^L)^n \) into \( L \). We say that \( \varphi \) is valid in \( V^L \) iff ZFC proves \( \| \varphi \| = 1 \).

FST is a first-order theory of logic BL∀ in the predicate language with binary predicates \( \in \) and =.

**Definition 6.2.** The following are axioms of the theory FST:

(extensionality) \( (\forall x)(\forall y)(x = y \equiv (\Delta(x \subseteq y) \& \Delta(y \subseteq x)) \)

(empty set) \( (\exists x)\Delta(x \rightarrow y \forall y \neg (y \in x)) \)

(pair) \( (\forall x)(\forall y)(\exists z)\Delta(\forall u)(u \in z \equiv (u = x \cup u = y)) \)

(union) \( (\forall x)(\exists z)\Delta(\forall u)(u \in z \equiv (\exists y)(u \in y \& y \in x)) \)

(weak power) \( (\forall x)(\exists z)\Delta(\forall u)(u \in z \equiv \Delta(u \subseteq x)) \)

(infinity) \( (\exists z)\Delta(\emptyset \in z \& (\forall x) \in z(x \cup \{x\} \in z)) \)
In FST one can define (in an obvious way) the predicates \textit{Crisp} (to be a crisp set) and \(\subseteq\) (to be subset) and \(HC\text{risp}\) (to be a hereditarily crisp set). \(HC\) is the class of all hereditarily crisp sets. We can also naturally define the notions of empty set, union, and singleton.

**Theorem 6.3.**

1. All logical axioms of \(BL\triangle\forall\) are valid in \(V^L\) and the formulas valid in \(V^L\) are closed under Modus Ponens and generalization.
2. The axioms of the theory FST are valid in \(V^L\).

The formulation of axioms is to be done carefully; e.g., if one deletes the Baaz delta from the axiom of extensionality, then the logic collapses to the classical logic.

**Theorem 6.4.** FST proves all axioms of ZF with all quantifiers restricted to \(HC\) (i.e., \(HC\) is an inner model of ZF (with its classical logic) in FST (with its logic \(BL\triangle\forall\)).

See [87] for all details. Admittedly, some ‘polishing’ remains to be done: possibly the axiom of choice is not necessary for showing that \(V^L\) is a model of FST. And apparently the composition of both interpretations is (equivalent to) the identical interpretation of \(ZF(C)\), resp. FST in itself (in the corresponding logic). Thus FST with added axiom of choice for hereditarily crisp sets (over \(BL\triangle\forall\)) turns out to be a conservative extension of ZFC (with its classical logic): you just add to ZFC new variables for fuzzy sets, postulate FST (and our fuzzy logic) for them and say that sets of ZFC are exactly all hereditarily crisp sets in the sense of FST.

### 6.3. Set theory with full comprehension

In this subsection we discuss the development of a Cantor fuzzy set theory, i.e., a theory with full comprehension: for each formula \(\varphi(x)\) we can prove the existence of a set \(\{x|\varphi(x)\}\) of all \(x\) satisfying \(\varphi(x)\). In classical logic this leads to Russell’s paradox: if \(a = \{x|\varphi(x)\}\) then \(a \subseteq a\) iff \(a \notin a\), which is a contradiction in classical logic. But this is also a contradiction in several fuzzy logics, notably in Gödel and product logic (in general, in logics whose negation is the so-called Gödel negation). But we have seen above that in Łukasiewicz logic it is well possible that a formula is equivalent to its own negation (its truth value is just \(\frac{1}{2}\)). Thus the question reads: can we have a set theory with full comprehension over Łukasiewicz set theory? Skolem in [133] gave a partial solution (a comprehension schema admitting Russell’s set \(\{x|x \notin x\}\)). His result was strengthened by Chang [26] and Fenstad [47]. Consistency of full comprehension with Łukasiewicz logic was finally proved by White [136]. Our question is whether we can develop some reasonable mathematics in this set theory. The whole section is a short survey of the results of [76], where the reader also finds more details and references.

Unfortunately, our main result is negative: if we introduce a set of natural numbers and postulate that it satisfies a simple schema of induction we get an inconsistency. This is because we shall be able to define, inside our set theory, a self-referring truth predicate \(\text{Tru}\) for natural numbers which provably commutes with connectives. This gives an inconsistency by Theorem 6.1.
Definition 6.5.

(1) The theory $\mathsf{CL}_0$ (weak Cantor-Łukasiewicz set theory) is a theory over $\mathsf{L}$ with single binary predicate $\in$; and (comprehension) terms $\{x | \varphi(x, \ldots)\}$ for each formula $\varphi(x, \ldots)$ (the occurrence of $x$ in comprehension terms is not considered free). $\mathsf{CL}_0$ has single (extralogical) axiom for each formula $\varphi(x, \ldots)$:

$$\forall u (u \in \{x | \varphi(x, \ldots)\} \equiv \varphi(u, \ldots))$$

(2) In $\mathsf{CL}_0$ we define (Leibniz or intentional) equality $x = y \equiv (\forall z)(x \in z \equiv y \in z)$.

$\mathsf{CL}_0$ proves crispness, reflexivity, symmetry and transitivity of equality; for each formula $\varphi(x, u, \ldots, v)$, $\mathsf{CL}_0$ proves congruence, i.e.,

$$(x = y \& \varphi(x, u, \ldots, v)) \rightarrow \varphi(y, u, \ldots, v).$$

In $\mathsf{CL}_0$ one defines the empty set and the usual operations of singleton, unordered and ordered pair, union, intersection and complement by some natural definitions. One also defines extensional equality: $x =_e y \equiv (\forall u)(u \in x \equiv u \in y)$. From the preceding we get provability of $x = y \rightarrow x =_e y$; but the converse is not provable; $\mathsf{CL}_0$ plus $x = y \equiv x =_e y$ is contradictory.

The theory proves a very powerful fixed point theorem, due to Cantini:

**Theorem 6.6.** For each formula $\varphi(x, \ldots, z)$, there is a term $\vartheta$ such that $\mathsf{CL}_0$ proves

$$\forall u (u \in \vartheta \equiv \varphi(u, \ldots, \vartheta)), \text{ i.e.}$$

$$\vartheta =_e \{u | \varphi(u, \ldots, \vartheta)\}.$$

(This is stronger than comprehension; $\vartheta$ may appear in $\varphi$.)

To introduce the set of all natural numbers using this theorem, we shall denote $\{x\}$ also as $S(x)$ (or $Sx$) and use it to define the successor of a natural number.

**Definition 6.7.** Using the fixed point theorem extend $\mathsf{CL}_0$ with a new constant $\omega$ and by $(\forall x)(x \in \omega \equiv (x = \emptyset \vee (\exists y \in \omega)(x = S(y))))$. Clearly, this is a conservative extension, denoted still $\mathsf{CL}_0$.

One can show that $\mathsf{CL}_0$ extended with the deduction rule

$$\frac{\varphi(0), (\forall x)(\varphi(x) \equiv \varphi(S(x)))}{(\forall x \in \omega)\varphi(x)}$$

for any formula $\varphi$ not containing the constant $\omega$ is consistent. But this deduction rule appears too weak, and allowing the rule for any $\varphi$ seems to be desirable and natural. Thus one defines as follows:

**Definition 6.8.** $\mathsf{CL}$ is the theory $\mathsf{CL}_0$ extended by the deduction rule

$$\frac{\varphi(0), (\forall x)(\varphi(x) \equiv \varphi(S(x)))}{(\forall x \in \omega)\varphi(x)}$$

for any formula $\varphi(x)$, possibly containing $\omega$ and having further free variables.

This seems at first to be a very nice theory; first, $\mathsf{CL}$ proves the crispness of $\omega$, i.e., $(\forall x)(x \in \omega \rightarrow x \in^2 \omega)$ (where $x \in^2 \omega$ is $x \in \omega \& x \in \omega$).

Second, using the above fixed point theorem one may introduce arithmetical operations of addition and multiplication and using the last deduction rule prove them to be crisp and to satisfy all axioms of Peano arithmetic $\mathsf{PA}$.

*But* one can go further and develop a sort of model theory and using the fixed point theorem define a truth predicate for the arithmetical language extended with the truth predicate itself; moreover one can prove that this truth predicate commutes with connectives—and this is the tragedy; as stated in the section on the liar paradox, it follows that our
theory CL (set theory over Łukasiewicz logic with full comprehension and the deduction rule of induction over natural numbers for all set-theoretical formulas) is contradictory. Let us repeat that all details of this are in the paper [76]. Also it is fair to mention the predecessor of [76], namely the paper [70], which correctly develops arithmetic in CL but erroneously claims that the theory is consistent (overlooking the fact that one can add to $\mathbf{CL}_0$ consistently only the instances of the schema for formulas not containing the constant $\omega$). It remains to be an open question if one can develop reasonable mathematics in some extension of $\mathbf{CL}_0$ which can be proved consistent (relative to ZF or so).

Note that the proof of the fact that the theory CL is inconsistent was radically simplified in Yatabe’s papers [138,139].

6.4. Further reading

There is an ongoing project of the Prague research group in fuzzy logic, directed towards developing the logic-based fuzzy mathematics, i.e., an ‘alternative’ mathematics built in a formal analogy with classical mathematics, but using a suitable (t-norm based) predicate fuzzy logic instead of the classical logic. The proposed foundational theory is (in [13]) called Fuzzy Class Theory (FCT $^1$) and it is a first-order theory over multi-sorted predicate t-norm based fuzzy logic, which can be seen as Henkin-style higher order fuzzy logic and has a very natural axiomatic system which approximates nicely Zadeh’s original notion of fuzzy set [140]. The papers written within the project so far can be divided into several groups:

- Methodological issues: [11,14,16]
- Formal development of FCT: [13,35] and freely available primer [15]
- Theory of fuzzy relations: [12,17]
- Fuzzy topology: [18,19]
- Fuzzy filters and measures: [104,105]
- Fuzzy algebra and (interval) analysis: [10,96]

Finally we would like to mention a (Church-style) fuzzy type theory developed by Novák [120] over the logic IMTL$^{\triangle}$ (and later generalized to other logics). It has been mainly used as a formal background for linguistic modeling, see e.g., [122,123].

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References


17 The name Fuzzy Class Theory is also use to refer to the project as such, see www.cs.cas.ca/fct.


