O-convergence of fuzzy nets and its applications

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Abstract

In this paper, an O-convergence theory of fuzzy nets is built in \(L\)-topological spaces by means of neighborhoods of fuzzy points. It has many nice properties. It can be used to characterize the closed set, open set, \(T_2\) separation axiom and fuzzy compactness.

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1. Introduction

Pu and Liu \cite{7} introduced the concept of Q-neighborhoods of a fuzzy point and built successfully a theory of Moore–Smith convergence of fuzzy nets. Adherence points and accumulation points of fuzzy sets can be characterized by means of fuzzy nets. In \cite{5} it was pointed out that \(L\)-convergence classes and \(L\)-topologies completely determined each other.

In this paper, O-convergence theory of fuzzy nets is presented in terms of neighborhoods of fuzzy points. Although it is not so ideal as the convergence theory in \cite{5}, it still has many nice properties. In particular we can characterize closed sets, open sets, \(T_2\) separation axiom and fuzzy compactness by means of O-convergence and O-cluster points of fuzzy nets. Hence we overcome the difficulty which the neighborhood method meets.

In 1976, the concept of fuzzy compactness was introduced in [0,1]-topological spaces by R. Lowen \cite{6}. Subsequently its characterization was given by G.J. Wang in terms of \(\alpha\)-net in \cite{10} and by Chadwick in term of filter in \cite{1}. In 1988, Wang extended Lowen’s fuzzy compactness into \(L\)-topology in terms of \(\alpha\)-nets in \cite{12}, where \(\alpha\) is a nonzero co-prime element in lattice value

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L of $L^X$. Wang’s definition of fuzzy compactness is not equivalent to Kubičk’s definition of fuzzy compactness in [4]. In fact, Kubičk’s definition of fuzzy compactness implies Wang’s definition of fuzzy compactness, but the inverse is not true. Up till now we have not known whether Kubičk’s definition of fuzzy compactness can be characterized by means of fuzzy nets. Since the set of all nonzero co-prime elements in a completely distributive lattice is $\vee$-generating, based on the action of $\alpha$-nets ($\alpha$ is a nonzero co-prime element), we need a completely distributive lattice $L$ in $L^X$. This assumption is not too strong. In fact if $L$ is a continuous lattice with a quasi-complementation, then $L^{op}$ is also a continuous lattice. Thus our assumption of completely distributivity is only added a distributive law, i.e. $a \land (b \lor c) = (a \land b) \lor (a \land c)$. As pointed out by Kubičk [4], for a continuous lattice $L$ and a topological space $(X,T)$, $T = t_{L} \omega_{L}(T)$ is not true in general, but by Proposition 3.5 in [4] we know that one sufficient condition of $T = t_{L} \omega_{L}(T)$ is that $L$ is completely distributive.

2. Preliminaries

Throughout this paper $(L, \vee, \wedge, ^{t})$ is a completely distributive lattice with an order-reversing involution ‘$^{t}$’. $X$ is a nonempty set. The smallest element and the largest element in $L$ are denoted by $0$ and $1$. $L^X$ is the set of all $L$-fuzzy sets (or $L$-set for short) on $X$. The smallest element and the largest element in $L^X$ are denoted by $0$ and $1$.

An element $a$ is called co-prime [2] if $a \leq b \lor c$ implies $a \leq b$ or $a \leq c$. The set of nonzero co-prime elements in $L$ is denoted by $M(L)$. Note that the set of all nonzero co-prime elements in $L^X$ are exactly the set of all $L$-fuzzy points $x_\lambda (\lambda \in M(L))$ defined by $x_\lambda(x) = \lambda$ and $x_\lambda(y) = 0$ otherwise. We shall not distinguish a crisp set and its character function.

An $L$-topological space ($L$-ts) is a pair $(L^X, \delta)$, where $\delta(\subseteq L^X)$ contains $0$ and $1$ and is closed for any suprema and finite infima. Members of $\delta$ are called open, and their quasi-complements are called closed. A closed $L$-set $P$ is called a closed remote-neighborhood (or closed R-neighborhood) of $x_\lambda$ if $x_\lambda \not\in P$. An open $L$-set $Q$ is called an open $Q$-neighborhood of $x_\lambda$ if $Q'$ is a closed $R$-neighborhood of $x_\lambda$. An open $L$-set $U$ is called an open neighborhood of $x_\lambda \in M(L^X)$ if $x_\lambda \in U$. All closed R-neighborhoods of $x_\lambda$ are denoted by $\eta^-(x_\lambda)$. All open Q-neighborhoods of $x_\lambda$ are denoted by $\eta^0(x_\lambda)$. All open neighborhoods of $x_\lambda$ are denoted by $\eta(x_\lambda)$.

$x_\lambda \in M(L^X)$ is said to be quasi-coincident with $B \subseteq L^X$ if $x_\lambda \not\in B'$.

As proved by Hutton [3], if $L$ is a completely distributive lattice and $a \in L$, then there exists $B \subseteq L$ such that:

1. $a = \bigvee B$, and
2. If $A \subseteq L$ and $a = \bigvee A$, then for each $b \in B$ there is a $c \in A$ such that $b \leq c$.

But for a complete lattice $L$, if $\forall a \in L$, there exists $B \subseteq L$ satisfying (1) and (2), then in general $L$ is not a completely distributive lattice. To turn $L$ into a completely distributive lattice, Wang revised condition (2) as follows.

(2') If $A \subseteq L$ and $a \leq \bigvee A$, then for each $b \in B$ there is a $c \in A$ such that $b \leq c$.

Wang proved that a complete lattice $L$ is completely distributive if and only if for each element $a$ in $L$, there exists $B \subseteq L$ satisfying (1) and (2'). Such a set $B$ is called a minimal set of $a$ by Wang [11]. Let $\beta(a)$ denote the union of all minimal set of $a$ and $\beta^0(a) = \beta(a) \cap M(L)$. One easily sees that both $\beta(a)$ and $\beta^0(a)$ are minimal set of $a$ (see [5,11]).
Proposition 2.1 (Gierz et al. [2], Liu and Luo [5], Wang [11]). If $L$ is a completely distributive lattice, then $M(L)$ is a join-generating set of $L$.

In a completely distributive lattice $L$, it is easy to see that $b \in \beta^*(a) \iff b \ll a$, where $\ll$ is the way below relation [2] in $L$.

Definition 2.2 (Shi [9]). Let $A \in L^X$ and $a \in L$, we define

$$A[a] = \{ x \in X \mid A(x) \geq a \}, \quad A(a) = \{ x \in X \mid a \in \beta(A(x)) \}, \quad A^{(a)} = \{ x \in X \mid A(x) \not\ll a \}.$$  

It is easy to prove that $A(0) = A^{(0)}$, $A(a) \subseteq A[a]$, $A^{(a)} = \bigcup_{b \not\ll a} A[b] = \bigcup_{b \not\ll a} A(b)$.

Definition 2.3 (Liu and Luo [5]). If $D$ is a directed set, then every mapping $S : D \rightarrow M(L^X)$ is called a net in $L^X$. A net $T : E \rightarrow M(L^X)$ is called a subnet of $S : D \rightarrow M(L^X)$ if there exists a mapping $N : E \rightarrow D$ such that

(i) $T = S \circ N$;

(ii) For each $n_0 \in D$, there exists $m_0 \in E$ such that $N(m) \geq n_0$ for $m \geq m_0$.

Since for a co-prime element $x$, $\beta^*(x)$ is a directed set, we know that $S = \{ x_r \mid r \in \beta^*(x) \}$ is a net in $L^X$.

Definition 2.4 (Liu and Luo [5]). An $L$-ts $(L^X, \delta)$ is called a $T_2$ space (or Hausdorff space) if for any $x, y \in M(L^X)$ with $x \neq y$, there exists $U \in \mathcal{P}(x)$, $V \in \mathcal{P}(y)$ such that $U \cap V = \emptyset$.

Definition 2.5 (Wang [12], Zhao [14]). Let $(L^X, \delta)$ be an $L$-ts and $B \in L^X$. $\mathcal{A} \subseteq \delta'$ is called an $\alpha$-R-neighborhood family of $B$, briefly $\alpha$-RF of $B$, if for each point $x \in B$, there exists $A \in \mathcal{A}$ such that $x \not\ll A$. $\mathcal{A}$ is called an $\alpha$-$\gamma$-RF of $B$, briefly $\alpha$-$\gamma$-RF of $B$, if there exists $\gamma \in \beta^*(x)$ such that $\mathcal{A}$ is a $\gamma$-RF of $B$, where $x \in M(L)$.


Definition 2.6 (Wang [12]). Let $(L^X, \delta)$ be an $L$-ts and $D \in L^X$, $D$ is called fuzzy compact if for all $x \in M(L)$ and for all $\gamma \in \beta^*(x)$, every constant $\alpha$-net in $D$ has a cluster point $x_\gamma \ll D$.

The following theorem gives two characterizations of Definition 2.6.

Theorem 2.7 (Shi [8]). Let $(L^X, \delta)$ be an $L$-ts and $D \in L^X$. Then the following statements are equivalent.

1. $D$ is fuzzy compact.
2. For all $x \in M(L)$, each $\alpha$-$\gamma$-RF of $D$ has a finite subfamily which is an $\alpha$-RF of $D$.
3. For all $x \in M(L)$, each $\alpha$-$\gamma$-RF of $D$ has a finite subfamily which is an $\alpha$-$\gamma$-RF of $D$.

Condition (3) in Theorem 2.7 is exactly the definition in [13].
Kubiak also extended Lowen fuzzy compactness into $L$-topology, but Kubiak’s definition is not equivalent to Definition 2.6. Moreover we have not known whether Kubiak’s fuzzy compactness can be characterized by means of fuzzy nets.

3. O-convergence of nets

**Definition 3.1.** A net $S$ with index set $D$ is also denoted by $\{S(n) \mid n \in D\}$ or $\{S(n)\}_{n \in D}$. For $A \subseteq L^{X}$, a net $S$ is said to quasi-coincide with $A$ if $\forall n \in D$, $S(n) \notin A'$.

**Definition 3.2.** Let $\{S(n) \mid n \in D\}$ be a net in $(L^{X}, \delta)$, $x_{\lambda} \in M(L^{X})$. $S$ eventually possesses the property $P$, if there exists $n_{0} \in D$ such that $\forall n \geq n_{0}$, $S(n)$ always possesses the property $P$. $S$ frequently possesses the property $P$, if for every $n \in D$, there always exists $n_{0} \in D$ such that $n_{0} \geq n$ and $S(n_{0})$ possesses the property $P$.

$x_{\lambda}$ is an O-cluster point of $S$, if $\forall U \in \mathcal{N}_{0}(x_{\lambda})$, $S$ is frequently in $U$. $x_{\lambda}$ is an O-limit point of $S$, if $\forall U \in \mathcal{N}_{0}(x_{\lambda})$, $S$ is eventually in $U$, in this case we also say that $S$ O-converges to $x_{\lambda}$, denoted by $S \xrightarrow{O} x_{\lambda}$.

It is easy to prove the following theorem.

**Theorem 3.3.** Let $S$ be a net in $(L^{X}, \delta)$, $T$ a subnet of $S$ and $x_{\lambda}, x_{\mu} \in M(L^{X})$. Then

1. $S = \{x_{r} \mid r \in \beta^{*}(\lambda)\}$ O-converges to $x_{\lambda}$.
2. $S \xrightarrow{O} x_{\lambda}$ implies that $x_{\lambda}$ is an O-cluster point of $S$.
3. If $x_{\lambda} \leq x_{\mu}$ and $x_{\lambda}$ is an O-cluster point of $S$, then $x_{\mu}$ is also an O-cluster point of $S$.
4. $S \xrightarrow{O} x_{\lambda} \leq x_{\mu} \Rightarrow S \xrightarrow{O} x_{\mu}$.
5. $S \xrightarrow{O} x_{\lambda} \Rightarrow T \xrightarrow{O} x_{\lambda}$.
6. $x_{\lambda}$ is an O-cluster point of $T \Rightarrow x_{\lambda}$ is an O-cluster point of $S$.
7. $x_{\lambda}$ is an O-cluster point of $S$ if and only if $S$ has a subnet $R$ such that $R \xrightarrow{O} x_{\lambda}$.

**Proof.** It is simple and is omitted. □

The following theorem characterizes the closure of an $L$-set.

**Theorem 3.4.** Let $x_{\lambda} \in M(L^{X})$, $B \subseteq L^{X}$. Then $x_{\lambda}$ quasi-coincides with $B^{-}$ if and only if there exists a net $S$ quasi-coinciding with $B$ such that $S \xrightarrow{O} x_{\lambda}$.

**Proof.** *Necessity.* Suppose that $x_{\lambda}$ quasi-coincides with $B^{-}$. Then $\forall U \in \mathcal{N}_{0}(x_{\lambda})$, $U \notin (B^{-})'$. Further $B^{-} \nsubseteq U'$. Hence $B \nsubseteq U'$. This implies $U \nsubseteq B'$. Take $S(U) \in M(L^{X})$ such that $S(U) \subseteq U$, $S(U) \notin B'$. We obtain a net $\{S(U) \mid U \in \mathcal{N}_{0}(x_{\lambda})\}$ O-converging to $x_{\lambda}$ and it quasi-coincides with $B$.

* Sufficiency. *Let $\{S(n)\}$ be a net quasi-coinciding with $B$ and $S \xrightarrow{O} x_{\lambda}$. If $x_{\lambda} \leq (B^{-})'$, then $\exists n_{0} \in D$ such that $\forall n \geq n_{0}$, $S(n) \leq (B^{-})' \leq B'$, contradicts that $S$ quasi-coincides with $B$. □
The following two corollaries are obvious.

**Corollary 3.5.** Let \((L^X, \delta)\) be an \(L\)-ts and \(A \subseteq L^X\). Then the following conditions are equivalent:

1. \(A\) is closed.
2. For each net \(S\) quasi-coinciding with \(A\), if \(x_1\) is an \(O\)-cluster point of \(S\), then \(x_1 \notin A'\).
3. For each net \(S\) quasi-coinciding with \(A\), if \(S \xrightarrow{O} x_1\), then \(x_1 \notin A'\).

**Corollary 3.6.** Let \((L^X, \delta)\) be an \(L\)-ts and \(A \subseteq L^X\). Then the following conditions are equivalent:

1. \(A\) is open.
2. \(\forall x_1 \leq A, S \xrightarrow{O} x_1\) implies \(S\) is eventually in \(A\).
3. \(\forall x_1 \leq A\), if \(x_1\) is \(O\)-cluster point of \(S\), then \(S\) is frequently in \(A\).

Moreover we can easily prove the following result.

**Theorem 3.7.** Let \(f : (L^X, \delta) \rightarrow (L^Y, \zeta)\) be an \(L\)-value Zadeh’s type mapping. Then the following conditions are equivalent.

1. \(f\) is continuous.
2. For any net \(S\) in \(L^X\), if \(x_1\) is an \(O\)-cluster point of \(S\), then \(f(x_1)\) is an \(O\)-cluster point of \(f(S)\).
3. For any net \(S\) in \(L^X\), if \(S \xrightarrow{O} x_1\), then \(f(S) \xrightarrow{O} f(x_1)\).

The following theorem gives a characterization of a \(T_2\) \(L\)-ts.

**Theorem 3.8.** An \(L\)-ts \((L^X, \delta)\) is \(T_2\) if and only if for any \(x, y \in M(L^X)\) with \(x \neq y\), there exist \(U \in \mathcal{N}^O(x_1), V \in \mathcal{N}^O(y_\mu)\) such that \(U \cap V = \emptyset\).

**Proof.** \((\Rightarrow)\) Suppose that \((L^X, \delta)\) is \(T_2\) and \(x, y \in M(L^X)\) with \(x \neq y\). Then for any co-prime element \(a \notin \lambda'\) and any co-prime element \(b \notin \mu'\), there exist \(U(a) \in \mathcal{P}(x_a), V(b) \in \mathcal{P}(y_b)\) such that \(U \cap V = \emptyset\). Let \(U = \bigvee_{a \notin \lambda'} U(a)\) and \(V = \bigvee_{b \notin \mu'} V(b)\), then \(U \in \mathcal{N}^O(x_1), V \in \mathcal{N}^O(y_\mu)\) and \(U \cap V = \emptyset\).

\((\Leftarrow)\) Suppose for any \(x, y \in M(L^X)\) with \(x \neq y\), there exist \(U \in \mathcal{N}^O(x_a), V \in \mathcal{N}^O(y_b)\) such that \(U \cap V = \emptyset\). To prove that \((L^X, \delta)\) is \(T_2\), suppose that \(x, y \in M(L^X)\) with \(x \neq y\). Then for any co-prime element \(a \notin \lambda'\) and any co-prime element \(b \notin \mu'\), there exist \(U(a) \in \mathcal{N}^O(x_a), V(b) \in \mathcal{N}^O(y_b)\) such that \(U \cap V = \emptyset\). Let \(U = \bigvee_{a \notin \lambda'} U(a)\) and \(V = \bigvee_{b \notin \mu'} V(b)\), then \(U \in \mathcal{P}(x_a), V \in \mathcal{P}(y_\mu)\) and \(U \cap V = \emptyset\). This shows that \((L^X, \delta)\) is \(T_2\). \(\square\)

A \(T_2\) \(L\)-ts can be characterized by \(O\)-convergence of nets.

**Theorem 3.9.** An \(L\)-ts \((L^X, \delta)\) is \(T_2\) if and only if for each net \(S\) in \(L^X\) having \(O\)-limit points, support points of \(O\)-limit points of \(S\) is unique.
Proof. Suppose that \((L^X, \delta)\) is \(T_2\) and \(x_\lambda, y_\mu\) are two O-limit points of a net \(S(n)\). If \(x \neq y\), then there exists \(U \in \mathcal{N}^0(x_\lambda), V \in \mathcal{N}^0(y_\mu)\) such that \(U \cap V = \emptyset\). Since \(x_\lambda, y_\mu\) are two O-limit points of \(S(n)\), we can take \(n_0\) such that when \(n \geq n_0\), \(S(n) \subseteq U\) and \(S(n) \subseteq V\), further \(S(n) \subseteq U \cap V\). But \(U \cap V = \emptyset\). This is a contradiction. Therefore it follows that \(x = y\).

Conversely suppose that support points of O-limit points of each net in \(L^X\) having O-limit points is unique. We prove that \((L^X, \delta)\) is \(T_2\). Suppose that it is not \(T_2\). Then there exists \(x_\lambda, y_\mu \in M(L^X)\) with \(x \neq y\) such that \(\forall U \in \mathcal{N}^0(x_\lambda), \forall V \in \mathcal{N}^0(y_\mu), U \cap V \neq \emptyset\). \(\forall U \in \mathcal{N}^0(x_\lambda), \forall V \in \mathcal{N}^0(y_\mu), \) take \(S(uv) \in M(L^X)\) such that \(S(uv) \subseteq U \cap V\) and let

\[ S = \{S(uv) | U \in \mathcal{N}^0(x_\lambda), V \in \mathcal{N}^0(y_\mu)\}. \]

\(\forall (U_1, V_1), (U_2, V_2) \in \mathcal{N}^0(x_\lambda) \times \mathcal{N}^0(y_\mu)\) we define \((U_1, V_1) \leq (U_2, V_2) \Leftrightarrow U_1 \supseteq U_2, V_1 \supseteq V_2\). Then \(\mathcal{N}^0(x_\lambda) \times \mathcal{N}^0(y_\mu)\) is a directed set. It is easy to see that \(S\) is a net O-converging to \(x_\lambda\) and \(y_\mu\). Thus we obtain that \(x = y\), this contradicts \(x \neq y\). The proof is obtained. \(\square\)

4. Characterizations of fuzzy compactness

Kubiak pointed out that his definition of fuzzy compactness cannot be restated in terms of open \(L\)-sets by simply applying the order-reversing involution on \(L^X\), since this involution need not be order-reversing with respect to the way-below relation. In this section, we shall present some characterizations of Wang’s definition of fuzzy compactness by means of open \(L\)-sets and O-cluster points. We first prove the following Lemma.

Lemma 4.1. For each \(a \in L - \{0\}\), let \(Q(a) = \{b \in L | b \not\leq a'\}\), \(Q^*(a) = Q(a) \cap M(L)\). Then the following conditions are true.

1. For each \(a \in L, \gamma \in \beta^*(a) \Rightarrow \beta^*(\bigwedge Q(\gamma)) \cap Q^*(a) \neq \emptyset\).
2. \(b \in \beta^*(\bigwedge Q(a)) \Rightarrow \bigwedge Q(b) \geq a\).

Proof. (1) For \(a \in L, \gamma \in \beta^*(a)\), we suppose that \(\beta^*(\bigwedge Q(\gamma)) \cap Q^*(a) = \emptyset\). Then \(\forall e \in \beta^*(\bigwedge Q(\gamma)), e \leq a'\). Hence \(\bigwedge Q(\gamma) \leq a'\), i.e. \(a \leq \bigvee\{c' | c \in Q(\gamma)\}\). Since \(\gamma \in \beta^*(a)\), there exists a \(c \in Q(\gamma)\) such that \(\gamma \leq c'\). This is a contradiction.

(2) Suppose that \(b \in \beta^*(\bigwedge Q(a))\). Then \(\forall c \not\leq a', b \leq c\). Hence \(\forall d \in Q(b), d \not\leq c'\). This implies that \(a \not\leq d\). Further we obtain that \(\bigwedge Q(b) \geq a\). \(\square\)

To characterize fuzzy compactness, the following definition is useful.

Definition 4.2. Let \((L^X, \delta)\) be an \(L\)-ts, \(G \subseteq L^X\) and \(x \in M(L)\). \(\Phi \subseteq \delta\) is called a \(Q_x\)-open cover of \(G\), if for each \(x_\lambda \not\leq G'\), it follows that \(x_\lambda \not\leq \bigvee \Phi = \bigvee \{A | A \in \Phi\}\).

It is obvious that \(\Phi\) is a \(Q_x\)-open cover of \(1\) if and only if \(\Phi\) is an open cover of constant fuzzy set \(x\). \(\Phi\) is a \(Q_x\)-open cover of \(G\) if and only if \(\Phi\) is an open cover of \(x \wedge G^{(\sigma')}\) if and only if \(G' \bigvee (\bigvee \Phi) \supseteq x\).
Theorem 4.3. Let \((L^X, \delta)\) be an L-ts and \(G \in L^X\). Then \(G\) is fuzzy compact if and only if \(\forall x \in M(L), \forall \gamma \in \beta^*(x), \) each \(Q_x\)-open cover \(\Phi\) of \(G\) has a finite subfamily \(B\) such that \(B\) is a \(Q_y\)-open cover of \(G\).

Proof. \((\Rightarrow)\) Suppose that \(G\) is fuzzy compact, \(x \in M(L)\) and \(\Phi\) is a \(Q_x\)-open cover of \(G\). For each \(\gamma \in \beta^*(x)\), take an \(a \in \beta^*(x)\) such that \(\gamma \in \beta^*(a)\). By Lemma 4.1(1) we can take \(b \in \beta^*(\bigwedge Q(\gamma) \cap Q^*(a))\). Take \(c \in \beta^*(b)\) such that \(c \in Q^*(a)\). Now we prove that \(\Phi'\) is a \(c\)-RF of \(G\).

In fact, suppose that \(\Phi'\) is not a \(c\)-RF of \(G\). Then there exists an \(x_c \subseteq G\) such that \(x_c \subseteq \bigwedge \Phi', \) i.e., \(c \not\subseteq \bigwedge \{A'(x) \mid A \in \Phi\}\). Hence \(a \not\subseteq \bigwedge \{A(x) \mid A \in \Phi\}\) and \(a \not\subseteq G(x)'\). This shows that \(\Phi\) is not a \(Q_x\)-open cover of \(G\). This contradicts that \(\Phi\) is a \(Q_x\)-open cover of \(G\).

Thus \(\Phi'\) is a \(b\)-RF of \(G\). Hence by fuzzy compactness of \(G\) we know that there exists a finite subfamily \(\Omega\) of \(\Phi\) such that \(\Omega'\) is a \(b\)-RF of \(G\). Now we shall prove that \(\Omega\) is a \(Q_x\)-open cover of \(G\).

Suppose that \(\Omega\) is not a \(Q_x\)-open cover of \(G\). Then there exists an \(x_c \subseteq G\) such that \(x_c \not\subseteq \bigwedge \Omega\). Hence \(\gamma \not\subseteq \bigwedge \Omega\) and \(A \in \Omega\), \(\gamma \not\subseteq A(x)\). We shall prove that \(\Omega\) is fuzzy compact. Let \(x \in M(L)\) and \(\Psi\) is an \(x\)-RF of \(G\). Then there exists a \(\gamma \in \beta^*(x)\) such that \(\Psi\) is a \(\gamma\)-RF of \(G\). Take an \(a \in \beta^*(x)\) such that \(\gamma \in \beta^*(a)\). By Lemma 4.1(1) we can take a \(b \in \beta^*(\bigwedge Q(\gamma) \cap Q^*(a))\). Take \(c \in \beta^*(b)\) such that \(c \in Q^*(a)\). Now we prove that \(\Psi'\) is a \(Q_x\)-open cover of \(G\).

In fact, suppose that \(\Psi'\) is not a \(Q_x\)-open cover of \(G\). Then there exists an \(x_b \subseteq G\) such that \(x_b \not\subseteq \bigwedge \Psi'\). Hence \(b \not\subseteq G'\) and \(A \in \Psi\), \(b \not\subseteq B'(x)\), i.e., \(G(x) \not\subseteq b'\) and \(B(x) \not\subseteq b'\). This implies that \(G(x) \in Q(b)\) and \(B(x) \in Q(b)\). By Lemma 4.1(2) we know that \(\bigwedge Q(b) \geq \gamma\). Therefore \(\gamma \not\subseteq G(x)\) and \(A \in \Xi\), \(\gamma \geq B(x)\). This contradicts that \(\Psi\) is a \(\gamma\)-RF of \(G\).

Thus \(\Psi'\) has a finite subfamily \(\Omega\) which is a \(Q_x\)-open cover of \(G\). Now we shall prove that \(\Omega\) is an \(x\)-RF of \(G\).

Suppose that \(\Omega\) is not an \(x\)-RF of \(G\). Then there exists an \(x_x \subseteq G\) such that \(x_x \not\subseteq \bigwedge \Omega'\). Hence \(x \not\subseteq G(x)\) and \(x \not\subseteq \bigwedge \{A'(x) \mid A \in \Omega\}\). Thus we obtain that \(x \not\subseteq G(x)\) and \(x \not\subseteq \bigwedge \{A'(x) \mid A \in \Omega\}\). Further \(c \not\subseteq G'(x)\) and \(c \not\subseteq \bigwedge \{A(x) \mid A \in \Omega\}\). This contradicts that \(\Omega\) is a \(Q_x\)-open cover of \(G\).

So we complete the proof. \(\square\)

From Theorem 4.3 we easily obtain the following two results.

Corollary 4.4. \((L^X, \delta)\) be an L-ts and \(G \in L^X\). Then \(G\) is fuzzy compact if and only if \(\forall x \in M(L), \forall \gamma \in \beta^*(x)\) and for each \(\Phi \subseteq \delta\) such that \(G' \cup (\bigvee \Phi) \supseteq x\), there exists a finite subfamily \(\Psi \subseteq \Phi\) such that \(G' \cup (\bigvee \Psi) \supseteq \gamma\).

Corollary 4.5. \((L^X, \delta)\) is fuzzy compact if and only if \(\forall x \in M(L), \forall \gamma \in \beta^*(x)\), each open cover \(\Phi\) of \(x\) has a finite subfamily \(B\) such that \(B\) is an open cover of \(x\).

Fuzzy compactness can also be characterized by means of O-cluster points of nets.
Definition 4.6. Let \( x \in M(L) \). A net \( \{ S(n) \mid n \in D \} \) in \( L^X \) is called an \( x^- \)-net if there exists \( n_0 \in D \) such that \( \forall n \geq n_0 \), \( V(S(n)) \leq \alpha \), where \( V(S(n)) \) denotes the height of \( S(n) \). A net \( \{ S(n) \} \) in \( L^X \) is said to be a constant \( x^- \)-net if the height of each \( S(n) \) is a constant value \( \alpha \).

Obviously each constant \( x^- \)-net must be an \( x^- \)-net.

Theorem 4.7. An \( L \)-set \( G \) is fuzzy compact in \( (L^X, \delta) \) if and only if \( \forall x \in M(L), \forall \gamma \in \beta^*(x) \), each constant \( \gamma^- \)-net quasi-coinciding with \( G \) has an \( O \)-cluster point \( x_s \) quasi-coinciding with \( G \).

Proof. Suppose that \( G \) is fuzzy compact. For \( x \in M(L) \) and \( \gamma \in \beta^*(x) \), let \( \{ S(n) \mid n \in D \} \) is a constant \( \gamma^- \)-net quasi-coinciding with \( G \). Suppose that \( S \) has no any \( O \)-cluster point \( x_s \) quasi-coinciding with \( G \). Then for each \( x_s \notin G \), there exist \( U_s \in M^\infty(x_s) \) and \( n_s \in D \) such that \( \forall n \geq n_s \), \( S(n) \notin U_s \). Take \( \Phi = \{ U_s \mid x_s \notin G \} \), then \( \Phi \) is a \( Q_s \)-open cover of \( G \). Since \( G \) is fuzzy compact, \( \Phi \) has a finite subfamily \( \Psi = \{ U_{i_1}, U_{i_2}, \ldots, U_{i_k} \} \) such that \( \Psi \) is a \( Q_{\gamma} \)-open cover of \( G \). Since \( D \) is a directed set, there exists \( n_0 \in D \) such that \( n_0 \geq n_{i_s} \) for each \( i \leq k \). Thus we can obtain that \( \forall n \geq n_0 \), \( S(n) \notin \bigvee \{ U_{i_s} \mid i = 1, 2, \ldots, k \} \). This contradicts that \( \Psi \) is a \( Q_{\gamma} \)-open cover of \( G \). Therefore \( S \) has at least an \( O \)-cluster point \( x_s \notin G \).

Conversely suppose that \( \forall x \in M(L), \forall \gamma \in \beta^*(x) \), each constant \( \gamma^- \)-net quasi-coinciding with \( G \) has an \( O \)-cluster point \( x_s \notin G \). We now prove that \( G \) is fuzzy compact. Let \( \Phi \) be a \( Q_s \)-open cover of \( G \). If there exists \( \gamma \in \beta^*(x) \) such that each finite subfamily \( \Psi \) of \( \Phi \) is not a \( Q_{\gamma} \)-open cover of \( G \), then for each finite subfamily \( \Psi \) of \( \Phi \), there exists \( S(\Psi) \in M(L^X) \) with height \( \gamma \) such that \( S(\Psi) \notin G \) and \( S(\Psi) \notin \bigvee \Psi \). Take \( S = \{ S(\Psi) \mid \Psi \) is a finite subfamily of \( \Phi \} \).

Then \( S \) is a constant \( \gamma^- \)-net quasi-coinciding with \( G \). By \( \gamma \in \beta^*(x) \) we can take \( s \in \beta^*(x) \) such that \( \gamma \in \beta^*(s) \). Then \( S \) has an \( O \)-cluster point \( x_s \) quasi-coinciding with \( G \). Hence for each finite subfamily \( \Psi \) of \( \Phi \) we have that \( x_s \notin \bigvee \Psi \), in particular, \( x_s \notin B \) for each \( B \in \Phi \). But since \( \Phi \) be a \( Q_s \)-open cover of \( G \), we know that there exists \( B \in \Phi \) such that \( x_s \leq B \), this is a contradiction. So \( G \) is fuzzy compact. \( \square \)

Theorem 4.8. An \( L \)-set \( G \) is fuzzy compact in \( (L^X, \delta) \) if and only if \( \forall x \in M(L), \forall \gamma \in \beta^*(x) \), each \( \gamma^- \)-net quasi-coinciding with \( G \) has an \( O \)-cluster point \( x_s \) quasi-coinciding with \( G \).

Proof. The sufficiency is obvious. We need only prove the necessity.

Let \( G \) be fuzzy compact, \( x \in M(L) \), \( \gamma \in \beta^*(x) \) and \( \{ S(n) \mid n \in D \} \) an \( \gamma^- \)-net quasi-coinciding with \( G \). Then there exists \( n_0 \in D \) such that \( \forall n \geq n_0 \), \( S(n) \leq \gamma \). Put \( E = \{ n \in D \mid n \geq n_0 \} \) and

\[
T = \{ T(n) \mid n \in E, V(T(n)) = \gamma, \text{support point of } T(n) \text{ is the same as } S(n) \}.
\]

Then \( T \) is a constant \( \gamma^- \)-net quasi-coinciding with \( G \). Let \( x_s \) be an \( O \)-cluster point of \( T \). It is easy to see that \( x_s \) is also an \( O \)-cluster point of \( S \). \( \square \)

Remark 4.9. Because Kubis\koppacute; definition of fuzzy compactness depends on each elements \( a \in L \), but the definition of nets in \([5,12]\) was only defined in \( M(L) \), we do not know whether Kubis\koppacute; definition of fuzzy compactness can be characterized by \( O \)-cluster points. We leave it as an open problem.
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