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A generalization of Tardiff's fixed point theorem in probabilistic metric spaces and applications to random equations

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Abstract

Using the infinitely countable extension of triangular norms, a generalization of Tardiff's fixed point theorem in probabilistic metric spaces is proved. As a consequence, an application to random equations is obtained.

Keywords: Probabilistic metric space; Triangular norm; Menger space; Iterative roots of the function; Fixed point theorem

1. Introduction and preliminaries

The theory of probabilistic metric spaces [17] was developed by many authors. The study of contraction mappings for probabilistic metric spaces was initiated by Sehgal, Sherwood and Bharucha-Reid [18–20]. Some further results on the existence of the fixed point of a q-probabilistic contraction can be found in [1,5–7,14–16].

We investigated in [5,7] the countable extension of t-norms and we introduced a new notion: the geometrically convergent (briefly g-convergent) t-norm, which is closely related to the fixed point theory. We proved that t-norms of H-type and some subclasses of Dombi, Aczél–Alsina, and Sugeno–Weber families of t-norms are geometrically convergent, see [7]. A new approach to the fixed point theory in probabilistic metric spaces is given in Tardiff's paper [21], where some additional growth conditions for the mapping $\mathcal{F} : S \times S \to \mathcal{D}^+$ are assumed under the condition $T \geqslant T_L$. V. Radu [13] introduced a

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stronger growth condition for $\mathcal{F}$ than in Tardiff's paper (under the condition $T \geqslant T_{\mathrm{L}}$, which enables him to define a metric) and by metric approach an estimation of the convergence with respect to the solution can be obtained, see [5].

Using the countable extension of triangular norms we prove in this paper, a generalization of Tardiff's fixed point theorem in probabilistic metric spaces. An application to random equations is given.

Let $\mathcal{D}^+$ be the set of all distribution functions F such that $F(0) = 0$ (F is a nondecreasing, left continuous mapping from $\mathbb{R}$ into [0, 1] such that $\sup_{x\in\mathbb{R}} F(x) = 1$).

The ordered pair $(S, \mathcal{F})$ is said to be *a probabilistic metric space* if S is a nonempty set and $\mathcal{F} : S \times S \to \mathcal{D}^+$ ($\mathcal{F}(p, q)$ is denoted by $F_{p,q}$ for every $(p, q) \in S \times S$) satisfies the following conditions:

(1) $F_{u,v}(x) = 1$ for every $x > 0 \Leftrightarrow u = v$ $(u, v \in S)$.

(2) $F_{u,v} = F_{v,u}$ for every $u, v \in S$.

(3) $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1 \Rightarrow F_{u,w}(x + y) = 1$ for $u, v, w \in S$ and $x, y \in \mathbb{R}^+$.

If only (1) and (2) from above holds, the ordered pair $(S, \mathcal{F})$ is said to be *a probabilistic semi-metric space*. A *Menger space* (see [17]) is a triple $(S, \mathcal{F}, T)$, where $(S, \mathcal{F})$ is a probabilistic metric space, T is a triangular norm (abbreviated t-norm) and the following inequality holds:

$$F_{u,v}(x + y) \geqslant T(F_{u,w}(x), F_{w,v}(y)) \text{ for every } u, v, w \in S \text{ and every } x > 0,\ y > 0.$$

Recall that a mapping $T : [0, 1] \times [0, 1] \to [0, 1]$ is called a *triangular norm* (a t-norm), see [10], if the following conditions are satisfied:

$T(a, 1) = a$ for every $a \in [0, 1]$, $T(a, b) = T(b, a)$ for every $a, b \in [0, 1]$,

$$a \geqslant b,\ c \geqslant d \quad \Rightarrow \quad T(a, c) \geqslant T(b, d) \quad (a, b, c, d \in [0, 1]),$$

$$T(a, T(b, c)) = T(T(a, b), c) \quad (a, b, c \in [0, 1]).$$

Example 1. The following are the three basic continuous t-norms:

(i) The minimum t-norm, T_{M}, is defined by

$$T_{\mathrm{M}}(x, y) = \min(x, y),$$

(ii) The product t-norm, T_{P}, is defined by

$$T_{\mathrm{P}}(x, y) = x \cdot y,$$

(iii) The Łukasiewicz t-norm T_{L} is defined by

$$T_{\mathrm{L}}(x, y) = \max(x + y - 1, 0).$$

As regards the pointwise ordering, we have the inequalities $T_{\mathrm{L}} < T_{\mathrm{P}} < T_{\mathrm{M}}$. Each t-norm T can be extended (by associativity) in a unique way to an n-ary operation taking for $(x_1, \ldots, x_n) \in [0, 1]^n$, $n \in \mathbb{N}$, the values $T(x_1, \ldots, x_n)$ which are defined by

$$\mathop{\mathrm{T}}_{i=1}^{0} x_i = 1, \quad \mathop{\mathrm{T}}_{i=1}^{n} x_i = T\left(\mathop{\mathrm{T}}_{i=1}^{n-1} x_i, x_n\right) = T(x_1, \ldots, x_n).$$

We have for two important t-norms T_{L} and T_{M} that

$$T_{\mathrm{L}}(x_1, \ldots, x_n) = \max\left(\sum_{i=1}^{n} x_i - (n - 1), 0\right)$$

and

$$T_{\mathrm{M}}(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n),$$

respectively.

A t-norm T can be extended to a countable infinitary operation taking for any sequence $(x_n)_{n\in\mathbb{N}}$ from $[0, 1]$ the values

$$\mathop{\mathsf{T}}_{i=1}^{\infty} x_i = \lim_{n\to\infty} \mathop{\mathsf{T}}_{i=1}^{n} x_i. \tag{1}$$

The limit on the right-hand side of (1) exists since the sequence $\left(\mathsf{T}_{i=1}^{n} x_i\right)_{n\in\mathbb{N}}$ is nonincreasing and bounded from below. Some sufficient conditions for T were given in [5,7] to ensure that $\lim_{n\to\infty} \mathsf{T}_{i=n}^{\infty} x_i = 1$.

The (ε, λ)-topology in $(S, \mathcal{F})$ is generated by the family of neighbourhoods

$$\mathcal{U} = (U_v(\varepsilon, \lambda))_{(v,\varepsilon,\lambda)\in S\times\mathbb{R}^+\times(0,1)}, \text{ where } \quad U_v(\varepsilon, \lambda) = \{u \mid u \in S, F_{u,v}(\varepsilon) > 1 - \lambda\}.$$

The (ε, λ)-uniformity in $(S, \mathcal{F})$ is given by the family $(U_{(\varepsilon,\lambda)})_{(\varepsilon,\lambda)\in\mathbb{R}^+\times(0,1)}$, where

$$U_{(\varepsilon,\lambda)} = \{(u, v) \mid u, v \in S, F_{u,v}(\varepsilon) > 1 - \lambda\},$$

and it exists (only) if $\sup_{x<1} T(x, x) = 1$, otherwise it is a semi-uniformity.

A sequence $(p_n)_{n\in\mathbb{N}}$ in a probabilistic metric space $(S, \mathcal{F})$ is an $\mathcal{F}$-Cauchy sequence if for every $\varepsilon > 0$ and $\lambda \in (0, 1)$ there exists $n_0(\varepsilon, \lambda) \in \mathbb{N}$ such that $F_{p_n,p_m}(\varepsilon) > 1 - \lambda$, for every $n, m \geqslant n_0(\varepsilon, \lambda)$. A probabilistic metric space $(S, \mathcal{F})$ is complete if every $\mathcal{F}$-Cauchy sequence converges in S.

Let (Ω, Σ, P) be a probability measure space, and (M, d) a complete separable metric space. Let S be the space of all classes $\hat{X}$ of equivalence of measurable mappings $X : \Omega \to M$, i.e., $X, Y \in \hat{X}$ if and only if $X = Y$ a.e.. Then $(S, \mathcal{F}, T_{\mathrm{L}})$ is a complete Menger space, where, for every $\hat{X}, \hat{Y} \in S$,

$$\mathcal{F}_{\hat{X},\hat{Y}}(\varepsilon) = P(\{\omega \mid \omega \in \Omega, d(X(\omega), Y(\omega)) < \varepsilon\}) \quad (\varepsilon > 0).$$

If a t-norm T is such that $\sup_{x<1} T(x, x) = 1$, then $(S, \mathcal{F}, T)$ is, with the (ε, λ) topology, a metrizable topological space.

As an extension of the Banach contraction principle the first theorem on the existence of the fixed point in a Menger space $(S, \mathcal{F}, T_{\mathrm{M}})$ was proved in [18,19].

Theorem A. *Let $(S, \mathcal{F}, T_{\mathrm{M}})$ be a complete Menger space and $f : S \to S$. If there exists $q \in (0, 1)$ such that for all points $x, y \in S$ and all $t > 0$*

$$F_{fx,fy}(qt) \geqslant F_{x,y}(t), \tag{2}$$

then there exists a unique globally attractive fixed point of f.

Generally, we shall write simply *fp* instead of $f(p)$. If $f : S \to S$ satisfies (2) then f is called *q-probabilistic contraction*.

Since 1972 many authors investigated the possibility of the weakening of the condition that t-norm T is equal to T_M, see [5]. If $T \geqslant T_L$, an interesting result is obtained by Tardiff in [21], where a growth condition for $\mathcal{F}$ is introduced.

Theorem B (*Tardiff [21]*). *Let $(S, \mathcal{F}, T)$ be a complete Menger space such that $T \geqslant T_L$, $f : S \to S$ a q-probabilistic contraction and there exists $x_0 \in S$ such that*

$$\int_1^{\infty} \ln u \, \mathrm{d}F_{x_0, fx_0}(u) < \infty.$$

Then there exists a unique fixed point of f.

A generalization of the notion of the q-probabilistic contraction [18], the so-called ψ-probabilistic contraction, is investigated by many authors [2–5,9] for single-valued and multi-valued mappings.

Definition 2. Let $(S, \mathcal{F})$ be a probabilistic metric space, $f : S \to S$ and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$. The mapping f is called a ψ-probabilistic contraction iff for every $x, y \in S$ and every $t > 0$

$$F_{fx,fy}(\psi(t)) \geqslant F_{x,y}(t).$$

If $\psi(t) = qt$, for every $t > 0$, where $q \in (0, 1)$, then a ψ-probabilistic contraction is a q-probabilistic contraction.

If T is a t-norm let for every $x \in [0, 1]$ and $n \in \mathbb{N}$

$$x_T^{(n)} = \begin{cases} 1, & n = 0, \\ T(x_T^{(n-1)}, x) & \text{otherwise.} \end{cases}$$

A t-norm T is of H-type if the family of functions $\{x \to x_T^{(n)}\}_{n \in \mathbb{N}}$ is equicontinuous at the point $x = 1$. By $\psi^{(n)}$ we shall denote the nth iteration of a mapping ψ $(n \in \mathbb{N})$.

In [9] a fixed point theorem for a multi-valued ψ-probabilistic contraction is proved. From Theorem 1 in [9] the next corollary follows.

Corollary 3. *Let $(S, \mathcal{F}, T)$ be a complete Menger space and T be of H-type.*

Let $f : S \to S$ be a ψ-probabilistic contraction and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be nondecreasing and such that

$$\sum_{n \in \mathbb{N}} \psi^{(n)}(t) < \infty \text{ for every } t > 0. \quad (3)$$

Then there exists a fixed point of the mapping f.

A similar result is obtained in [3] where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a strictly increasing continuous function with $\psi(0) = 0$ and $\psi(t) < t$, for every $t > 0$.

Remark 4. In Lemma 2.1 [3] it is necessary to suppose that $\psi(\mathbb{R}^+) = \mathbb{R}^+$ since in the proof the existence of $\psi^{-1} : \mathbb{R}^+ \to \mathbb{R}^+$ is supposed.

In [3] the continuity of ψ is used in order to prove that for every $t > 0$

$$\lim_{n\to\infty} \psi^{(n)}(t) = 0 \tag{4}$$

$$\lim_{n\to\infty} (\psi^{-1})^{(n)}(t) = \infty. \tag{5}$$

In [6] the following result is proved.

Theorem C. *Let* $(S, \mathcal{F}, T)$ *be a complete Menger space,* $f : S \to S$ *be a* ψ*-probabilistic contraction and* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a strictly increasing bijection such that* (4) *holds. If there exists* $x_0 \in S$ *such that*

$$\lim_{n\to\infty} \mathop{\mathrm{T}}_{i=n}^{\infty} F_{x_0, f x_0}(\psi^{-(i)}(\bar{t})) = 1 \tag{6}$$

for some $\bar{t} > 0$, *then there exists a unique fixed point* $\overline{x} = \lim_{n\to\infty} f^n x_0$ *of the mapping* f.

Remark 5. If t-norm T is of H-type the condition (6) is satisfied for every $x_0 \in S$.

Remark 6. If in Theorem C, $\psi(t) = qt$ $(t > 0)$, where $q \in (0, 1)$ and $T \geqslant T_{\mathrm{L}}$ condition (6) is satisfied if

$$\int_1^{\infty} \ln u \, \mathrm{d}F_{x_0, f x_0}(u) < \infty \tag{7}$$

since (7) implies that by [21]

$$\lim_{n\to\infty} \left(\mathop{T_{\mathrm{L}}}_{i=n}^{\infty} \right) F_{x_0, f x_0} \left(\frac{t}{q^n} \right) = 1 \quad \text{for every } t > 0$$

and so

$$\lim_{n\to\infty} \mathop{\mathrm{T}}_{i=n}^{\infty} F_{x_0, f x_0} \left(\frac{t}{q^n} \right) \geqslant \lim_{n\to\infty} \left(\mathop{T_{\mathrm{L}}}_{i=n}^{\infty} \right) F_{x_0, f x_0} \left(\frac{t}{q^n} \right) = 1.$$

We prove in this paper a fixed point theorem for a class of ψ-probabilistic contractions where $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is a bijection such that (3) and (5) hold.

2. Iterative roots of a given function

In the proof of the fixed point theorem for ψ-probabilistic contraction, we shall use an iterative root of the mapping $\psi : \mathbb{R}^+ \to \mathbb{R}^+$, i.e., a mapping $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$(\alpha \circ \alpha)(t) = \alpha^{(2)}(t) = \psi(t) \;\text{ for every } t > 0. \tag{8}$$

There is a large literature on iterative roots of a bijection [8,11,12]. Here, we recall a result of Lojasiewicz [12] on the existence of an iterative root of a bijection.

Let X be an arbitrary nonempty set and $\psi : X \to X$ be a bijection. By L_k $(k \in \mathbb{N} \cup \{0\})$ the cardinality of the set of k-cycles of the mapping ψ is denoted. Let $d_0 = n$ and $d_k = n/n_k$ $(k \in \mathbb{N})$, where n_k is the greatest divisor of n which is relative prime to k.

Theorem L (*Lojasiewicz [12]*). *Let $\psi : X \to X$ be a bijection. The iterative functional equation*

$$\alpha^{(n)}(t) = \psi(t) \quad \text{for every } t \in X \tag{9}$$

has a solution $\alpha : X \to X$ iff for every $k \in \mathbb{N} \cup \{0\}$, L_k is infinite or L_k is divisible by d_k.

If in (9) $n = 2$ then $d_0 = 2$ and $d_k = 2/n_k$ $(k \in \mathbb{N})$, where n_k is the greatest divisor of 2 which is relative prime to k. Hence

$$d_{2n} = 2,\ d_{2n-1} = 1, \quad n \in \mathbb{N}.$$

From Theorem **L** the next corollary follows.

Corollary 7. *Let $\psi : X \to X$ be a bijection. Then the iterative functional equation*

$$\alpha^{(2)}(t) = \psi(t) \quad \text{for every } t \in X \tag{10}$$

has a solution $\alpha : X \to X$ iff for every $k \in \mathbb{N} \cup \{0\}$, L_k is infinite or L_k is divisible by 2 for even k.

Corollary 8. *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a bijection such that* (4) *holds. Then the iterative functional equation*

$$\alpha^{(2)}(t) = \psi(t) \quad \text{for every } t > 0 \tag{11}$$

has a solution $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$.

Proof. From (4) it follows that for every $k \in \mathbb{N}$ the set of k-cycles is empty. Hence

$$L_k = 0 \quad \text{for every } k \in \mathbb{N}$$

and by Corollary 7 there exists a solution α of iterative functional equation (11), i.e., α is an iterative root of the mapping ψ. □

Lemma 9. *Let $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ be a bijection such that* (3) *holds. Then every iterative root $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the condition*

$$\sum_n \alpha^{(n)}(t) < \infty \quad \text{for every } t > 0. \tag{12}$$

Proof. From (3) it follows that (4) is satisfied and by Corollary 8 there is an iterative root α of the mapping ψ. On the other hand for $n > 1$ and $t > 0$

$$\begin{aligned}\sum_n \alpha^{(n)}(t) &= \sum_n \alpha^{(2n)}(t) + \sum_n \alpha^{(2n+1)}(t)\\ &= \sum_n \psi^{(n)}(t) + \sum_n \psi^{(n)}(\alpha(t))\end{aligned}$$

and (3) implies (12). □

Lemma 10. *Let* $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ *be a bijection such that* (5) *holds.*
Let $H : \mathbb{R}^+ \to [0,1]$ *be such that* $\lim_{t\to\infty} H(t) = 1$ *and*

$$\lim_{n\to\infty} \mathop{\mathrm{T}}_{i=n}^{\infty} H(\psi^{-(i)}(t)) = 1 \quad \text{for every } t > 0. \tag{13}$$

Then

$$\lim_{n\to\infty} \mathop{\mathrm{T}}_{i=n}^{\infty} H(\alpha^{-(i)}(t)) = 1 \quad \text{for every } t > 0. \tag{14}$$

Proof. From (13) it follows that $\sup_{x<1} T(x,x) = 1$ which implies that $\lim_{(x,y)\to(1,1)} T(x,y) = 1$. Since

$$\begin{aligned}\mathop{\mathrm{T}}_{i=n}^{\infty} H(\alpha^{-(i)}(t)) &\geqslant T\left(\mathop{\mathrm{T}}_{\substack{i=i_0\\2i_0\geqslant n}}^{\infty} H(\alpha^{-(2i)}(t)), \mathop{\mathrm{T}}_{\substack{i=j_0\\2j_0+1\geqslant n}}^{\infty} H(\alpha^{-(2i+1)}(t))\right)\\ &\geqslant T\left(\mathop{\mathrm{T}}_{i=[n/2]}^{\infty} H(\psi^{-(i)}(t)), \mathop{\mathrm{T}}_{i=[n-1/2]}^{\infty} H(\psi^{-(i)}(\alpha^{-1}(t)))\right),\end{aligned}$$

(13) implies (14). □

By $\mathcal{M}$ we shall denote the class of all bijections $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that (3) and (4) holds. It is obvious that the function $\psi(t) = qt$ $(t > 0)$, where $q \in (0,1)$, belongs to the class $\mathcal{M}$.

3. A fixed point theorem for ψ-probabilistic contractions

Theorem 11. *Let* $(S, \mathcal{F}, T)$ *be a complete Menger space,* $\psi \in \mathcal{M}$ *and* $f : S \to S$ *be a* ψ*-probabilistic contraction. If there exists* $x_0 \in S$ *such that for every* $t > 0$

$$\lim_{n\to\infty} \mathop{\mathrm{T}}_{i=n}^{\infty} F_{x_0, fx_0}(\psi^{-(i)}(t)) = 1, \tag{15}$$

then there exists a unique fixed point $\overline{x}$ *of the mapping f and* $\overline{x} = \lim_{n\to\infty} f^n x_0$.

Proof. Since (15) implies that $\sup_{x<1} T(x,x) = 1$ the (ε,λ)-topology of S is metrizable. We shall prove that f is an uniformly continuous mapping. Let

$$N(\varepsilon,\lambda) = \{(u,v) | (u,v) \in S\times S,\ F_{u,v}(\varepsilon) > 1-\lambda\},$$

where $\varepsilon > 0$, $\lambda \in (0,1)$. The family $\mathcal{N} = \{N(\varepsilon,\lambda)\}_{\substack{\varepsilon>0\\ \lambda\in(0,1)}}$ defines the (ε,λ)-uniformity of S. It suffices to prove that for every $\eta > 0$ and $\lambda \in (0,1)$ there exists $\varepsilon > 0$ and $\zeta \in (0,1)$ such that

$$(\forall (x,y) \in S\times S) \quad (x,y) \in N(\varepsilon,\zeta) \to (fx, fy) \in N(\eta,\lambda). \tag{16}$$

Let $\eta > 0$ and $\lambda \in (0,1)$ be given. Since $\lim_{n\to\infty} \psi^{(n)}(t) = 0$, for every $t > 0$, there exists $\varepsilon > 0$ such that $\psi(\varepsilon) < \eta$. Then $F_{fx,fy}(\eta) \geqslant F_{fx,fy}(\psi(\varepsilon)) \geqslant F_{x,y}(\varepsilon)$ and $F_{x,y}(\varepsilon) > 1-\lambda$ implies that $F_{fx,fy}(\eta) > 1-\lambda$. Hence (16) holds for $\zeta = \lambda$.

Let $x_{n+1} = fx_n$, for every $n \in \mathbb{N} \cup \{0\}$. We shall prove that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, i.e., that for every $\varepsilon > 0$ and $\lambda \in (0,1)$ there exists $n_0(\varepsilon,\lambda) \in \mathbb{N}$ such that

$$F_{x_{n+m},x_n}(\varepsilon) > 1-\lambda \text{ for every } n \geqslant n_0(\varepsilon,\lambda) \text{ and every } m \in \mathbb{N}. \tag{17}$$

Let $\varepsilon > 0$ and $\lambda \in (0,1)$ be given. Since $\psi \in \mathcal{M}$ there exists an iterative root $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ of the mapping ψ. By Lemma 9, $\sum_n \alpha^{(n)}(t) < \infty$ for every $t > 0$. Let t be a fixed number from $\mathbb{R}^+$. Since $\sum_n \alpha^{(n)}(t) < \infty$, there exists $n_1(\varepsilon) \in \mathbb{N}$ such that $\sum_{i\geqslant n_1(\varepsilon)} \alpha^{(i)}(t) < \varepsilon$. Hence, for every $n \geqslant n_1(\varepsilon)$ and every $m \in \mathbb{N}$

$$\begin{aligned}
F_{x_{n+m},x_n}(\varepsilon) &\geqslant F_{x_{n+m},x_n}\left(\sum_{i\geqslant n_1(\varepsilon)} \alpha^{(i)}(t)\right) \\
&\geqslant F_{x_{n+m},x_n}\left(\sum_{i=n}^{n+m-1} \alpha^{(i)}(t)\right) \\
&\geqslant \underbrace{T(T(\ldots T}_{(m-1)\text{-times}}(F_{x_n,x_{n+1}}(\alpha^{(n)}(t)), F_{x_{n+1},x_{n+2}}(\alpha^{(n+1)}(t))), \\
&\quad \ldots, F_{x_{n+m-1},x_{n+m}}(\alpha^{(n+m-1)}(t))) \\
&\geqslant \underbrace{T(T(\ldots T}_{(m-1)\text{-times}}(F_{x_0,fx_0}(\psi^{-(n)}(\alpha^{(n)}(t))), F_{x_0,fx_0}(\psi^{-(n+1)}(\alpha^{(n+1)}(t))), \\
&\quad \ldots, F_{x_0,fx_0}(\psi^{-(n+m-1)}(\alpha^{(n+m-1)}(t)))) \\
&= \underbrace{T(T(\ldots T}_{(m-1)\text{-times}}(F_{x_0,fx_0}(\alpha^{-(n)}(t)), F_{x_0,fx_0}(\alpha^{-(n+1)}(t))), \\
&\quad \ldots, F_{x_0,fx_0}(\alpha^{-(n+m-1)}(t))) \\
&\geqslant \mathop{\mathbf{T}}_{i=n}^{\infty} F_{x_0,fx_0}(\alpha^{-(i)}(t)).
\end{aligned}$$

Applying Lemma 10 for $H(t) = F_{x_0,fx_0}(t)$, since (15) holds, we have that

$$\lim_{n\to\infty} \mathop{\mathbf{T}}_{i=n}^{\infty} F_{x_0,fx_0}(\alpha^{-(i)}(t)) = 1.$$

Let $n_2(\lambda) \in \mathbb{N}$ be such that for every $n \geqslant n_2(\lambda)$

$$\mathop{\mathbf{T}}_{i=n}^{\infty} F_{x_0,fx_0}(\alpha^{-(i)}(t)) > 1-\lambda.$$

Then (17) holds for $n_0(\varepsilon,\lambda) = \max\{n_1(\varepsilon),\ n_2(\lambda)\}$. The space S is complete and there exists $\overline{x} = \lim_{n\to\infty} x_n = \lim_{n\to\infty} f^n x_0$. Since f is continuous, $\overline{x}$ is a fixed point of the mapping f. Suppose that

$y \in S,\ \ y = fy$. Then for every $\varepsilon > 0$

$$F_{\overline{x},y}(\varepsilon) = F_{f\overline{x},fy}(\varepsilon) \geqslant F_{\overline{x},y}(\psi^{-1}(\varepsilon)) \geqslant \cdots \geqslant F_{\overline{x},y}(\psi^{-(n)}(\varepsilon))$$

and so from (4) it follows that

$$F_{\overline{x},y}(\varepsilon) = 1 \ \text{ for every } \varepsilon > 0.$$

This implies that $\overline{x} = y$. □

The family $(T_\lambda^{\mathrm{SW}})_{\lambda\in(-1,\infty]}$ of *Sugeno–Weber* t-*norms*, see [5,10], which contains T_{L}, is given by

$$T_\lambda^{\mathrm{SW}}(x,y) = \begin{cases} T_{\mathrm{P}}(x,y) & \text{if } \lambda = \infty, \\ \max\left(0, \dfrac{x+y-1+\lambda xy}{1+\lambda}\right) & \text{otherwise.} \end{cases} \tag{18}$$

Corollary 12. *Let $(S, \mathcal{F}, T)$ be a complete Menger space and there exists a* t-*norm $T_1 \in \bigcup_{\lambda\in(-1,\infty)}\{T_\lambda^{\mathrm{SW}}\}$ such that $T \geqslant T_1$. Let $\psi \in \mathcal{M}$ and $f : S \to S$ be a ψ-probabilistic contraction. If $\sum_n 1/\psi^{-(n)}(t) < \infty$, for every $t > 0$ and for some $x_0 \in S$*

$$\sup_{s>0} s(1 - F_{x_0,fx_0}(s)) < \infty,$$

then there exists a unique fixed point $\overline{x}$ of the mapping f and $\overline{x} = \lim_{n\to\infty} f^n x_0$.

Proof. Let $M > 0$ be such that

$$s(1 - F_{x_0,fx_0}(s)) \leqslant M \ \text{ for every } \ s > 0,$$

i.e.,

$$F_{x_0,fx_0}(s) > 1 - \frac{M}{s} \ \text{ for every } s > 0.$$

Let $s_0 > 0$ be such that $1 - M/s_0 > 0$. Then for every $s \geqslant s_0,\ F_{x_0,fx_0}(s) > 1 - M/s > 0$. Let $t > 0$. If $n_0(t) \in \mathbb{N}$ is such that

$$\psi^{-(n)}(t) \geqslant s_0 \ \text{ for } \ n \geqslant n_0(t),$$

then for every $n \geqslant n_0(t)$

$$F_{x_0,fx_0}(\psi^{-(n)}(t)) > 1 - \frac{M}{\psi^{-(n)}(t)} > 0.$$

Since $T_1 \in \bigcup_{\lambda\in(-1,\infty)}\{T_\lambda^{\mathrm{SW}}\}$ we have by [5] the equivalence

$$\sum_n \frac{M}{\psi^{-(n)}(t)} < \infty \Longleftrightarrow \lim_{n\to\infty} \left(\mathop{T_1}_{i=n}^{\infty}\right) \left(1 - \frac{M}{\psi^{-(i)}(t)}\right) = 1.$$

Hence

$$\lim_{n\to\infty} \mathop{\mathsf{T}}_{i=n}^{\infty} \left(1 - \frac{M}{\psi^{-(i)}(t)}\right) \geqslant \lim_{n\to\infty} \mathop{(T_1)}_{i=n}^{\infty} \left(1 - \frac{M}{\psi^{-(i)}(t)}\right) = 1$$

and so

$$\lim_{n\to\infty} \mathop{\mathsf{T}}_{i=n}^{\infty} F_{x_0, fx_0}(\psi^{-(i)}(t)) \geqslant \lim_{n\to\infty} \mathop{\mathsf{T}}_{i=n}^{\infty} \left(1 - \frac{M}{\psi^{-(i)}(t)}\right) = 1.$$

Hence, the relation (15) holds and we obtain the conclusion by Theorem 11. □

Now, by using Corollary 12, we can prove the following random fixed point theorem.

Theorem 13. *Let (Ω, Σ, P) be a probability measure space, (M, d) a complete separable metric space and $f : \Omega \times M \to M$ a continuous random operator.*

Suppose that there exists a mapping $\psi \in \mathcal{M}$ such that $\sum_n 1/\psi^{-(n)}(t) < \infty$ for every $t > 0$ and the following conditions hold:

(a) *For every $\hat{X}, \hat{Y} \in S$ and every $\varepsilon > 0$*

$$P(\{\omega | \omega \in \Omega, d(f(\omega, X(\omega)), f(\omega, Y(\omega))) < \psi(\varepsilon)\}) \geqslant P(\{\omega | \omega \in \Omega, d(X(\omega), Y(\omega)) < \varepsilon\})$$

for some $q \in (0, 1)$.

(b) *There exists $\hat{X}_0 \in S$ such that*

$$\sup_{u>0} u \ P(\{\omega | \omega \in \Omega, d(X_0(\omega), f(\omega, X_0(\omega))) \geqslant u\}) < \infty.$$

Then there exists a measurable mapping $X : \Omega \to M$ such that $X(\omega) = f(\omega, X(\omega))$ a.e.

Proof. The mapping $\hat{f} : S \to S$, defined by $(\hat{f}\hat{X})(\omega) = f(\omega, X(\omega))$ $(\omega \in \Omega)$ satisfies all the conditions of Corollary 12 for $T = T_{\mathrm{L}}$. □

Remark 14. Condition (b) in Theorem 13 is satisfied if $E(d(X_0(\omega), f(\omega, X_0(\omega))) < \infty$ since

$$P(\{\omega \mid \omega \in \Omega, d(X_0(\omega), f(\omega, X_0(\omega))) \geqslant u\}) \leqslant \frac{E(d(X_0(\omega), f(\omega, X_0(\omega)))}{u}, \quad u > 0,$$

holds.

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