A fixed point theorem for convex and decreasing operators

Ke Li, Jin Liang, Ti-Jun Xiao*

Department of Mathematics, University of Science and Technology of China, Hefei, Anhui 230026,
People’s Republic of China

Abstract

In this paper, we present a new fixed point theorem for noncompact, convex and decreasing operators, which extends the existing corresponding results. As a sample, we give an application of the fixed point theorem to the two-point boundary value problem for a second-order differential equation. © 2005 Elsevier Ltd. All rights reserved.

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1. Introduction

The fixed point theory for noncompact increasing or decreasing operators with convexity has been investigated extensively in the past 30 years and is applied to the study of various nonlinear equations (cf., e.g., [1–9] and the references therein). Stimulated by these works, in this paper, we will study further the existence and uniqueness of fixed points for convex and decreasing operators without compactness or continuity assumptions on the operators concerned, by using the partial-order method, the cone theory and the iterative technique. The new fixed point theorem obtained in this paper generalizes the relevant ones in [8,9] (our conditions are much weaker than those in [8,9]). Moreover, as a sample of application,
we applied our fixed point theorem to the following two-point boundary value problem for second-order differential equations:

\[
\begin{align*}
\begin{cases}
f(x) - \frac{d^2x}{dt^2} &= 0, & x \in [0, +\infty), & t \in [0, 1], \\
x(0) &= x(1) = 0,
\end{cases}
\end{align*}
\]

and obtain a new result on the existence and uniqueness of positive solution of problem (1), which is not a consequence of the corresponding fixed point theorems in [8,9].

Throughout this paper, \( \mathbb{N} \) denotes the set of natural numbers, \( E \) is a real Banach space with norm \( \| \cdot \| \), \( \theta \) denotes the zero element in \( E \), and \( P \) is a cone in \( E \). A partial order in \( E \) is given by \( x \geq y \) if and only if \( x - y \in P \).

**Definition 1.1.** \( P \) is called a normal cone if there exists a constant \( N \) such that

\[
\theta \leq x \leq y \implies \|x\| \leq N\|y\|,
\]

where \( N \) is called a normal constant.

**Definition 1.2.** Let \( D \) be a convex subset in \( E \). An operator \( A : D \to E \) is called a convex operator if

\[
A(tx + (1 - t)y) \leq tAx + (1 - t)Ay
\]

for all \( x, y \in D, x \leq y \) and \( t \in [0, 1] \).

**Definition 1.3.** Let \( D \subset E \). An operator \( A : D \to E \) is said to be a decreasing operator if

\[
x_1 \geq x_2 \implies Ax_1 \leq Ax_2,
\]

where \( x_1, x_2 \in D \).

2. Fixed point theorem

**Theorem 2.1.** Assume that \( P \) is a normal cone with \( N \) the normal constant, \( A : P \to P \) is a convex and decreasing operator, and \( A\theta > \theta \). If there exist \( \varepsilon \in (0, 1) \) and \( n_0, m_0 \in \mathbb{N} \cup \{0\} \) with \( n_0 > m_0 \) such that

(i) \( A^{2m_0+2}\theta - A^{2m_0}\theta \geq \varepsilon(A^{2m_0+3}\theta - A^{2m_0}\theta) \),

(ii) \( A^{2m_0}\theta \geq \frac{1}{2}(A^{2m_0+1}\theta + A^{2m_0}\theta) \),

then \( A \) has a unique fixed point \( x^* \) in \( P \).

Moreover, constructing successively a sequence \( x_n = Ax_{n-1} \) (\( n \in \mathbb{N} \)) for any initial value \( x_0 \in P \), we have

\[
\|x_n - x^*\| \to 0 \quad \text{as} \ n \to +\infty,
\]
and the rates of convergence are
\[
\|x_{2(m_0+n)} - x^*\| < \frac{N^2}{n - n_0 + m_0} \|A\| \quad (n > n_0 - m_0),
\]
\[
\|x_{2(m_0+n)+1} - x^*\| < \frac{N^2}{n - n_0 + m_0 + 1} \|A\| \quad (n > n_0 - m_0 - 1).
\]

Proof. Let \(u_0 = \theta, u_n = Au_{n-1}\) for each \(n \in \mathbb{N}\). By (i) and (ii), we get
\[
\begin{align*}
\tag{6}
u_{2(m_0+1)} & \geq \varepsilon u_{2m_0+3} + (1 - \varepsilon)u_{2m_0}, \\
2u_{2n} - u_{2m_0} & \geq u_{2m_0+1}. 
\tag{7}
\end{align*}
\]

By the decreasing property of \(A\), we have, for every \(n \in \mathbb{N}\),
\[
\theta = u_0 \leq u_2 \leq \cdots \leq u_{2m_0} \leq \cdots \leq u_{2(m_0+n)} \leq \cdots \leq u_{2(m_0+n)+1} \leq \cdots \leq u_{2m_0+3} \\
\leq \cdots \leq u_3 \leq u_1 = A\theta. 
\tag{8}
\]

Thus, it follows from (6) and (8) that
\[
\begin{align*}
\theta \leq \varepsilon(u_{2(m_0+n)+1}-u_{2m_0}) & \leq \cdots \leq \varepsilon(u_{2m_0+3}-u_{2m_0}) \\
\leq u_{2(m_0+1)} - u_{2m_0} & \leq \cdots \leq u_{2(m_0+n)} - u_{2m_0}.
\end{align*}
\]

For each \(n \in \mathbb{N}\), we set
\[
\alpha_n = \sup\{x > 0; u_{2(m_0+n)} \geq xu_{2(m_0+n)+1} + (1 - x)u_{2m_0}\},
\]
\[\beta_n = 1 - \alpha_n.\]

Obviously, for every \(n \in \mathbb{N}\),
\[
u_{2(m_0+n)} \geq \alpha_n u_{2(m_0+n)+1} + \beta_n u_{2m_0},
\]
\[0 < \varepsilon \leq \alpha_1 \leq \alpha_2 \leq \cdots \leq \alpha_n \leq \cdots \leq 1, \quad 0 \leq \beta_n < 1 - \varepsilon. \tag{9}\]

Next, we prove that \(\alpha_n \to 1\) as \(n \to +\infty\).

For each \(n \geq n_0 - m_0 - 1\), making use of inequalities (7)–(9) and the fact that \(A\) is convex and decreasing (i.e., (2) and (3) hold), we obtain
\[
\begin{align*}
u_{2(m_0+n)+1} & = Au_{2(m_0+n)} \leq A(\alpha_n u_{2(m_0+n)+1} + \beta_n u_{2m_0}) \\
& \leq \alpha_n u_{2(m_0+n)+1} + \beta_n u_{2m_0+1} \\
& \leq \alpha_n u_{2(m_0+n)+1} + \beta_n (2u_{2m_0} - u_{2m_0}) \\
& \leq (1 + \beta_n)u_{2(m_0+n)+1} - \beta_n u_{2m_0}.
\end{align*}
\]

Hence, for every \(n \geq n_0 - m_0 - 1\),
\[
u_{2(m_0+n)+1} \geq \frac{1}{1 + \beta_n} u_{2(m_0+n)+1} + \frac{\beta_n}{1 + \beta_n} u_{2m_0}.
\]
Thus, we see from the properties of $A$ and (7) that for all $n \geq n_0 - m_0 - 1$,

$$u_{2(m_0 + n + 1) + 1} = Au_{2(m_0 + n + 1)} \leq \frac{1}{1 + \beta_n} u_{2(m_0 + n + 1)} + \frac{\beta_n}{1 + \beta_n} u_{2m_0 + 1}$$

$$\leq \frac{1}{1 + \beta_n} u_{2(m_0 + n + 1)} + \frac{\beta_n}{1 + \beta_n} (2u_{2n_0} - u_{2m_0})$$

$$\leq \frac{1 + 2\beta_n}{1 + \beta_n} u_{2(m_0 + n + 1)} - \frac{\beta_n}{1 + \beta_n} u_{2m_0}.$$

So

$$u_{2(m_0 + n + 1)} \geq \frac{1 + \beta_n}{1 + 2\beta_n} u_{2(m_0 + n + 1) + 1} + \frac{\beta_n}{1 + 2\beta_n} u_{2m_0}.$$

This means that for all $n \geq n_0 - m_0 - 1$,

$$\alpha_{n+1} \geq \frac{1 + \beta_n}{1 + 2\beta_n},$$

that is,

$$\beta_{n+1} \leq \frac{\beta_n}{1 + 2\beta_n} = \frac{1}{2 + 1/\beta_n}.$$  

This implies that for every $n \geq n_0 - m_0$,

$$\beta_{n+1} \leq \frac{1}{4 + 1/(\beta_{n-1})} \leq \ldots \leq \frac{1}{2(n - n_0 + m_0) + 1/\varepsilon}$$

$$\leq \frac{1}{2(n - n_0 + m_0) + 1/(1 - \varepsilon)}$$

$$\leq \frac{1}{2(n - n_0 + m_0)}.$$  

Hence, $\beta_n \to 0$ as $n \to +\infty$, i.e., $x_n \to 1$ as $n \to +\infty$.

In addition, from (9), we deduce that for all $n, p \in \mathbb{N},$

$$0 < u_{2(m_0 + n + p)} - u_{2(m_0 + n)} \leq u_{2(m_0 + n) + 1} - u_{2(m_0 + n)}$$

$$\leq \beta_n (u_{2(m_0 + n) + 1} - u_{2m_0}) \leq \beta_n u_1.$$  

Since $P$ is a normal cone, $\{u_{2(m_0 + n)}\}_{n=1}^{\infty}$ is a Cauchy sequence. Hence there exists $x^* \in P$ such that $u_{2n} \to x^*$ as $n \to +\infty$. In combination with (11) and (8), we have

$$\lim_{n \to +\infty} u_{2n+1} = \lim_{n \to +\infty} u_{2n} = x^*,$$

$$u_{2(m_0 + n)} \leq x^* \leq u_{2(m_0 + n) + 1}.$$  

Therefore,

$$u_{2(m_0 + n) + 1} \leq Ax^* \leq u_{2(m_0 + n) + 1}.$$

Letting $n \to +\infty$ and using (12), we conclude that $Ax^* = x^*$, i.e., $x^*$ is a fixed point of $A$ in $P$. 

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Now, for any initial value $x_0 \in P$, consider the sequence

$$x_n = Ax_{n-1} \quad \text{for } n \in \mathbb{N},$$

and we get inductively,

$$u_{2(m_0+n)} \leq x_2(m_0+n) \leq u_{2(m_0+n)} - 1,$$

(14)

$$u_{2(m_0+n)} \leq x_2(m_0+n) + 1 \leq u_{2(m_0+n)} + 1.$$

(15)

By (12), we obtain

$$\|x_n - x^*\| \to 0 \quad \text{as } n \to +\infty.$$

Suppose that $y^* \in P$, $y^* \neq x^*$ and $Ay^* = y^*$, and let $x_0 = y^*$. Then

$$x_n = Ax_{n-1} = A^n x_0 = A^n y^* = y^*.$$

By (13), we get $y^* = x^*$. This implies the uniqueness of the fixed point of $A$ in $P$.

Finally, by (14) and the fact that $P$ is a normal cone, we obtain the following inequality:

$$\|x_2(m_0+n) - x^*\| \leq \|u_{2(m_0+n)} - x_2(m_0+n)\| + \|u_2(m_0+n) - x^*\| \leq 2N\|u_{2(m_0+n)-1} - u_2(m_0+n)\| \quad \text{for all } n > n_0 - m_0.$$

By (8) and (9), it is clear that

$$u_{2(m_0+n)} \geq u_{2(m_0+n-1)} \geq \alpha_{n-1} u_{2(m_0+n)-1} + \beta_{n-1} u_{2m_0}.$$

Therefore,

$$\theta > u_{2(m_0+n)-1} - u_{2(m_0+n)} \leq \beta_{n-1} (u_{2(m_0+n)-1} - u_{2m_0}) \leq \beta_{n-1} A\theta.$$

Thus, for every $n > n_0 - m_0$, we have

$$\|x^* - x_2(m_0+n)\| \leq 2N^2 \beta_{n-1} \|A\theta\| \leq \frac{N^2}{n - n_0 + m_0} \|A\theta\|,$$

where (10) was used. So (4) holds.

Arguing similarly as above, we can obtain (5) from (15). This ends the proof. \(\square\)

**Remark 2.2.** In Theorem 2.1, we do not need $A$ to be compact or continuous.

**Remark 2.3.** Theorem 2.1 extends the following result in [8]:

Let $E$ be a Banach space, $P$ a normal cone, and $A : P \to P$ a convex and decreasing operator. If there exist $\mu > 0$ and $m \in \mathbb{N}$ such that

$$A\theta > \theta, \quad \mu A\theta \leq A^2 \theta \quad \text{and} \quad \frac{1}{2} A\theta \leq A^{2m} \theta,$$

(16)

then $A$ has a unique fixed point $x^*$ in $P$. Moreover, for any initial value $x_0 \in [\theta, A\theta]$ and iterative sequence

$$x_n = Ax_{n-1} \quad (n \in \mathbb{N}),$$

we have $x_n \to x^*$ as $n \to +\infty$. 


In fact, for $m_0=0$, conditions (i) and (ii) of Theorem 2.1 read $A^2\theta \geq \varepsilon A^3 \theta$ and $A^{n_0} \theta \geq \frac{1}{2} A \theta$. These conditions are weaker than (16). Moreover, if the fixed point $x^* < \frac{1}{2} A \theta$, then the above fixed point theorem in [8], in contrast with Theorem 2.1, is not applicable.

3. An application

In this section, we apply Theorem 2.1 to the two-point boundary value problem (1) for second-order differential equations.

**Theorem 3.1.** Let function $f(x)$ satisfy the following conditions:

(i) $f(x)$ is a convex and decreasing function,

(ii) $f(0) = 1$, $0 < f\left(\frac{1}{8}\right) < 1$,

(iii) $f(\gamma;x) \geq \frac{1}{2}$ for all $x \in [0, \frac{1}{8}]$, where $\gamma = 1 - \frac{1}{8} f\left(\frac{1}{8}\right) [1 - f\left(\frac{1}{2}\right)]$.

Then problem (1.1) has a unique positive solution $x^*(t)$.

Moreover, $0 \leq x^*(t) \leq \frac{1}{2} t (1 - t)$ for all $t \in [0, 1]$.

**Proof.** It is well known that the solution (in $C[0, 1]$) of problem (1) is equivalent to the solution (in $C[0, 1]$) of the following Hammerstein integral equation:

$$x(t) = \int_0^1 G(t, s) f(x(s)) \, ds = (Ax)(t), \quad t \in [0, 1],$$

where $G(t, s)$ is the Green function of differential operator $-d^2/dt^2$ with boundary condition $x(0) = x(1) = 0$, i.e.,

$$G(t, s) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ t(1-s), & 0 \leq t \leq s \leq 1. \end{cases}$$

Define

$$P = \{ x \in C[0, 1] : x(t) \geq 0, t \in [0, 1] \}.$$

Clearly, $P$ is a normal cone in $C[0, 1]$. By (i), it is easy to show that operator $A : P \to P$ is convex and decreasing. Moreover,

$$(A\theta)(t) = \int_0^1 G(t, s) f(0) \, ds = \int_0^1 G(t, s) \, ds = \frac{1}{2} t (1 - t) > \theta(t).$$

This implies that $0 \leq (A\theta)(t) \leq \frac{1}{8}$. Hence

$$f\left(\frac{1}{8}\right) \leq f((A\theta)(t)) \leq 1. \quad (17)$$

Based on (17), we have

$$(A^2\theta)(t) = \int_0^1 G(t, s) f((A\theta)(s)) \, ds \geq f\left(\frac{1}{8}\right) (A\theta)(t). \quad (18)$$
On the other hand,

\[
(A^2 \theta)(t) = \int_0^1 G(t, s) f ((A \theta)(s)) \, ds \\
= (1 - t) \int_0^t s f \left( \frac{s(1 - s)}{2} \right) \, ds + t \int_t^1 (1 - s) f \left( \frac{s(1 - s)}{2} \right) \, ds.
\]

The fact that \(f(x)\) is convex yields that

\[
I_1 = (1 - t) \int_0^t s f \left( \frac{s(1 - s)}{2} \right) \, ds \\
\leq (1 - t) \int_0^t s(1 - s) f \left( \frac{s}{2} \right) \, ds + (1 - t) \int_0^t s^2 \, ds \\
\leq f \left( \frac{1}{2} \right) (1 - t) \int_0^t s^2(1 - s) \, ds + (1 - t) \int_0^t s(1 - s)^2 \, ds + \frac{t^3(1 - t)}{3} \\
= \frac{1}{6} \left[ f \left( \frac{1}{2} \right) (4t^2 - 3t^3) + 6t - 4t^2 + 3t^3 \right] (A \theta)(t).
\]

Similarly, we get

\[
I_2 = t \int_t^1 (1 - s) f \left( \frac{s(1 - s)}{2} \right) \, ds \\
= \frac{1}{6} \left[ f \left( \frac{1}{2} \right) (1 + t - 5t^2 + 3t^3) + 5 - 7t + 5t^2 - 3t^3 \right] (A \theta)(t).
\]

Hence,

\[
(A^2 \theta)(t) = I_1 + I_2 \leq \frac{1}{6} t [(1 - f \left( \frac{1}{2} \right)) t (t - 1) + 5 + f \left( \frac{1}{2} \right)] (A \theta)(t) \\
\leq \frac{5}{6} [5 + f \left( \frac{1}{2} \right)] (A \theta)(t). \tag{19}
\]

Moreover, (17) implies that

\[
f \left( f \left( \frac{1}{8} \right) (A \theta)(t) \right) \geq f ((A^2 \theta)(t)). \tag{20}
\]

Thus, it follows from (19) and (20) that

\[
(A^3 \theta)(t) = \int_0^1 G(t, s) f ((A^2 \theta)(s)) \, ds \\
\leq \int_0^1 G(t, s) f \left( \frac{1}{8} \right) (A \theta)(s) \, ds \\
\leq f \left( \frac{1}{8} \right) (A^2 \theta)(t) + \left[ 1 - f \left( \frac{1}{8} \right) \right] (A \theta)(t) \\
\leq \frac{1}{6} \left[ 5 + f \left( \frac{1}{2} \right) \right] f \left( \frac{1}{8} \right) (A \theta)(t) + \left[ 1 - f \left( \frac{1}{8} \right) \right] (A \theta)(t) \\
= \gamma (A \theta)(t).
\]
Therefore, 
\[ f((A^3\theta)(t)) \geq f(\gamma(A\theta)(t)). \]

In combination with (iii), we get
\[ (A^4\theta)(t) = \int_0^1 G(t, s) f((A^3\theta)(s)) \, ds \geq \frac{1}{2} \int_0^1 G(t, s) \, ds = \frac{1}{2} (A\theta)(t). \]

Letting \( \varepsilon = f(\frac{1}{8}) \) and using (18), we have
\[ (A^2\theta)(t) \geq \varepsilon(A\theta)(t) \geq \varepsilon(A^3\theta)(t). \]

Thus applying Theorem 2.1 with \( m_0 = 0, n_0 = 2 \) gives rise to the conclusion. \( \square \)

**Remark 3.2.** There exist a lot of functions satisfying (i)–(iii) in Theorem 3.1. For example, all the conditions are satisfied for \( f(x) = \frac{2}{2 + 17x} \), where \( \gamma = \frac{1943}{2079} \).

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**References**